The $C$ operator in $\mathcal{PT}$-symmetric quantum field theory transforms as a Lorentz scalar

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A non-Hermitian Hamiltonian has a real positive spectrum and exhibits unitary time evolution if the Hamiltonian possesses an unbroken $\mathcal{PT}$ (space-time reflection) symmetry. The proof of unitarity requires the construction of a linear operator called $C$. It is shown here that $C$ is the complex extension of the intrinsic parity operator and that the $C$ operator transforms under the Lorentz group as a scalar.

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I. INTRODUCTION

If the Hamiltonian that defines a quantum theory is Hermitian, $H = H^\dagger$, then we can be sure that the energy spectrum of the Hamiltonian is real and that the time evolution operator $U = e^{iHt}$ is unitary (probability preserving). (The symbol $\dagger$ represents conventional Dirac Hermitian conjugation; that is, combined complex conjugation and matrix transpose.) However, a non-Hermitian Hamiltonian can also have these desired properties. For example, the quantum-mechanical Hamiltonians

$$H = p^2 + x^2 + i\epsilon x^3$$ (1)

have been shown to have real positive discrete spectra [1,2] and to exhibit unitary time evolution [3]. Spectral positivity and unitarity are related to a crucial symmetry property of the Hamiltonian; namely, $\mathcal{PT}$ symmetry. (The linear operator $\mathcal{P}$ represents parity reflection and the antilinear operator $\mathcal{T}$ represents time reversal.) To summarize, if a Hamiltonian has an unbroken $\mathcal{PT}$ symmetry, then the energy levels are real and the theory is unitary.

The proof of unitarity requires the construction of a new linear operator called $C$, which in quantum mechanics is a sum over the eigenstates of the Hamiltonian [3]. The $C$ operator is then used to define the inner product for the Hilbert space of state vectors: $\langle A|B \rangle = A^{\mathcal{CPT}} \cdot B$. With respect to this inner product, the time evolution of the theory is unitary.

In the context of quantum mechanics, there have been a number of papers published on the calculation of $C$. A perturbative calculation of $C$ in powers of $\epsilon$ for the cubic Hamiltonian

$$H = p^2 + x^2 + i\epsilon x^3$$ (2)

was performed [4] and a perturbative calculation of $C$ for the two- and three-degree-of-freedom Hamiltonians

$$H = p^2 + q^2 + x^2 + y^2 + i\epsilon xy^2$$ (3)

and

$$H = p^2 + q^2 + r^2 + x^2 + y^2 + z^2 + i\epsilon xyz$$ (4)

was also performed [5]. A nonperturbative WKB calculation of $C$ for the quantum-mechanical Hamiltonians in (1) was done [6]. In quantum mechanics it was found that the $C$ operator has the general form

$$C = e^{i\mathcal{Q}}\mathcal{P},$$ (5)

where $\mathcal{Q}$ is a function of the dynamical variables (the coordinate and momentum operators). The simplest way to calculate $C$ in non-Hermitian quantum mechanics is to solve the three operator equations that define $C$:

$$C^2 = 1, \quad [C, \mathcal{PT}] = 0, \quad [C, H] = 0.$$ (6)

Following successful calculations of the $C$ operator in quantum mechanics, some calculations of the $C$ operator in field-theoretic models were done. A leading-order perturbative calculation of $C$ for the self-interacting cubic theory whose Hamiltonian density is

$$\mathcal{H}(x, t) = \frac{1}{2} \pi^2(x, t) + \frac{1}{2} \mu^2 \phi^2(x, t) + \frac{1}{2}[\nabla_x \phi(x, t)]^2 + i\epsilon \phi^4(x, t)$$ (7)

was performed [7,8] and the $C$ operator was also calculated for a $\mathcal{PT}$-symmetric version of quantum electrodynamics [9]. For each of these field theory calculations, it was assumed that the form for the $C$ operator is that given in (5), where $\mathcal{P}$ is now the field-theoretic version of the parity reflection operator; that is, for a scalar field

$$\mathcal{P} \phi(x, t) \mathcal{P} = \phi(-x, t)$$ (8)

and for a pseudoscalar field

$$\mathcal{P} \phi(x, t) \mathcal{P} = -\phi(-x, t).$$ (9)

However, in a recent investigation of the Lee model in which we calculated the $C$ operator exactly [10], we found that the correct field-theoretic form for the $C$ operator is not $C = e^{i\mathcal{Q}}\mathcal{P}$ but rather

$$C = e^{i\mathcal{Q}}\mathcal{P}_I.$$ (10)

Here, $\mathcal{P}_I$ is the intrinsic parity operator; $\mathcal{P}_I$ has the same effect on the fields as $\mathcal{P}$ except that it does not change the sign of the spatial argument of the fields. Thus, for a scalar field

$$\mathcal{P}_I \phi(x, t) \mathcal{P}_I = \phi(x, t).$$ (11)
and for a pseudoscalar field
\[ \mathcal{P}_I \varphi(x, t) \mathcal{P}_I = - \varphi(x, t). \tag{12} \]

Accordingly, the defining equations for \( C \) in (6) now become
\[ C^2 = 1, \quad [C, \mathcal{P}_I T] = 0, \quad [C, H] = 0. \tag{13} \]

The fundamental difference between the conventional parity operator \( \mathcal{P} \) and the intrinsic parity operator \( \mathcal{P}_I \) is revealed in their Lorentz transformation properties. For a quantum field theory that has parity symmetry the intrinsic parity operator \( \mathcal{P}_I \) is a Lorentz scalar because \( \mathcal{P}_I \) commutes with the generators of the homogeneous Lorentz group:
\[ [\mathcal{P}_I, J^{\mu \nu}] = 0. \tag{14} \]

However, the conventional parity operator \( \mathcal{P} \) does not commute with a Lorentz boost,
\[ [\mathcal{P}, J^{0i}] = -2f^{0i} \mathcal{P}, \tag{15} \]
and \( \mathcal{P} \) transforms as an infinite-dimensional reducible representation of the Lorentz group \([11]\). Specifically, \( \mathcal{P} \) transforms as an infinite direct sum of finite-dimensional tensors:
\[ (0, 1) \oplus (0, 3) \oplus (0, 5) \oplus (0, 7) \oplus \cdots. \tag{16} \]

That is, \( \mathcal{P} \) transforms as a scalar plus the spin-0 component of a two-index symmetric traceless tensor plus the spin-0 component of a four-index symmetric traceless tensor plus the spin-0 component of a six-index symmetric traceless tensor, and so on. Note that in (16) we use the notation of Ref. [12]. It was shown in Ref. [11] that the decomposition in (16) corresponds to a completeness summation over Wilson polynomials \([11, 13]\).

We believe that the correct way to represent the \( C \) operator is in (10) and not in (5). In the case of quantum mechanics there is, of course, no difference between these two representations because in this case \( \mathcal{P} = \mathcal{P}_I \). However, in quantum field theory, where \( \mathcal{P} \neq \mathcal{P}_I \), these two representations are different. For the case of the Lee model, (5) is the wrong representation for \( C \) and (10) is the correct representation. It is most remarkable that for the case of the cubic quantum field theory in (7) in either representation the functional \( Q \) is exactly the same. However, the representation of \( C \) in (10) is strongly preferred because, as we will show in this paper, it transforms as a Lorentz scalar \([14]\).

The work in this paper indicates for the first time the physical and mathematical interpretation of the \( C \) operator: The \( C \) operator is the complex analytic continuation of the intrinsic parity operator. To understand this remark, consider the Hamiltonian \( H \) associated with the Hamiltonian density \( \mathcal{H} \) in (7), \( H = \int dx \mathcal{H}(x, t) \). When \( \epsilon = 0 \), \( H \) commutes with \( \mathcal{P}_I \) and \( \mathcal{P}_I \) transforms as a Lorentz scalar. When \( \epsilon \neq 0 \), \( H \) does not commute with \( \mathcal{P}_I \), but \( H \) does commute with \( \mathcal{C} \). Moreover, we will see that \( C \) transforms as a Lorentz scalar. Furthermore, since \( Q \to 0 \) as \( \epsilon \to 0 \), we see that \( C \to \mathcal{P}_I \) in this limit. Therefore, we can interpret the \( C \) operator as the complex extension of the intrinsic parity operator.

This paper is organized very simply: In Sec. II we examine an exactly solvable \( \mathcal{P} \mathcal{T} \)-symmetric model quantum field theory and we calculate the \( C \) operator exactly. We show that this \( C \) operator transforms as a Lorentz scalar. Next, in Sec. III we examine the cubic quantum theory whose Hamiltonian density is given in (7) and show that the \( C \) operator, which was calculated to leading order in \( \epsilon \) in Ref. [7], again transforms as a Lorentz scalar. We make some concluding remarks in Sec. IV.

II. THE \( C \) OPERATOR FOR A TOY MODEL QUANTUM FIELD THEORY

In this section we calculate the \( C \) operator for an exactly solvable non-Hermitian \( \mathcal{P} \mathcal{T} \)-symmetric quantum field theory and show that \( C \) transforms as a Lorentz scalar. Consider the theory defined by the Hamiltonian density
\[ \mathcal{H}(x, t) = \mathcal{H}_0(x, t) + \epsilon \mathcal{H}_1(x, t), \tag{17} \]
where
\[ \mathcal{H}_0(x, t) = \frac{1}{2} \pi^2(x, t) + \frac{1}{2} \mu^2 \varphi^2(x, t) + \frac{1}{2} [\nabla_\mu \varphi(x, t)]^2, \]
\[ \mathcal{H}_1(x, t) = i \varphi(x, t). \tag{18} \]

We assume that the field \( \varphi(x, t) \) in (18) is a pseudoscalar. Thus, under intrinsic parity reflection \( \varphi(x, t) \) transforms as in (12). Also, \( \pi(x, t) = \bar{\varphi}(x, t) \), which is the field that is dynamically conjugate to \( \varphi(x, t) \), transforms as
\[ \mathcal{P}_I \pi(x, t) \mathcal{P}_I = - \pi(x, t). \tag{19} \]

To calculate the \( C \) operator we assume that \( C \) has the form in (10). The operator \( Q[\pi, \varphi] \), which is a functional of the fields \( \varphi \) and \( \pi \), can be expressed as a series in powers of \( \epsilon \):
\[ Q = \epsilon Q_1 + \epsilon^3 Q_3 + \epsilon^5 Q_5 + \cdots. \tag{20} \]

Note that only odd powers appear in the expansion of \( Q \). As shown in Ref. [7], we obtain \( Q_1 \) by solving the operator equation
\[ \left[ Q_1, \int dx \mathcal{H}_0(x, t) \right] = 2 \int dx \mathcal{H}_1(x, t). \tag{21} \]

To solve (21) we recall from Ref. [7] that the functional \( Q[\pi, \varphi] \) is odd in \( \pi \) and even in \( \varphi \). We then substitute the elementary ansatz
\[ Q_1 = \int dx R(x) \pi(x, t), \tag{22} \]
where \( R(x) \) is a c-number. We find that \( R(x) \) satisfies the functional equation

\[ \int dx R(x) \pi(x, t) = \int dx R(x) \pi(x, t), \tag{23} \]
\[ \int d\mathbf{x} \varphi(\mathbf{x}, t)(\mu^2 - \nabla_\perp^2)R(\mathbf{x}) = -2 \int d\mathbf{x} \varphi(\mathbf{x}, t), \quad (23) \]

whose solution is a constant:

\[ R(\mathbf{x}) = -2/\mu^2. \quad (24) \]

All higher-order contributions to \( Q[\pi, \varphi] \) in (20) vanish because

\[ [Q_1, [Q_1, H_1]] = [Q_1, -2V/\mu^2] = 0, \quad (25) \]

where \( V \) is the volume of the space. Therefore, the sequence of equations in Eq. (34) of Ref. \[7\] terminates after the first equation. We thus obtain the exact result \[15\]

\[ C = \exp\left(-\frac{2e}{\mu^2} \int d\mathbf{x} \pi(\mathbf{x}, t)\right) P_t. \quad (26) \]

The \( C \) operator is clearly a rotational scalar. To prove that \( C \) transforms as a Lorentz scalar, we must show that \( C \) commutes with the Lorentz boost operator

\[ J^{0i} = J_0^{0i} + e J_1^{0i}, \quad (27) \]

where

\[
\begin{align*}
J_0^{0i}(t) &= t \int d\mathbf{x} \pi(\mathbf{x}, t) \nabla_x \varphi(\mathbf{x}, t) - \int d\mathbf{x} x^i \mathcal{H}_0(\mathbf{x}, t), \\
J_1^{0i}(t) &= -\int d\mathbf{x} x^i \mathcal{H}_1(\mathbf{x}, t). \quad (28)
\end{align*}
\]

In the calculation above we assume that the integral \( \int d\mathbf{x} x^i \) vanishes by oddness, and in the last step we use the identity

\[
\left[ \int d\mathbf{x} x^i \varphi(\mathbf{x}, t), \int d\mathbf{y} \pi(\mathbf{y}, t) \right] = i \int d\mathbf{x} d\mathbf{y} x^i \delta(\mathbf{x} - \mathbf{y}) = i \int d\mathbf{x} x^i = 0. \quad (33) \]

Plugging (32) into the commutator (29), we obtain

\[ [C, J^{0i}] = 0, \quad (34) \]

which verifies that the \( C \) operator for this toy model transforms as a Lorentz scalar.

### III. THE \( C \) OPERATOR FOR AN \( i\varphi^3 \) QUANTUM FIELD THEORY

We consider next the cubic quantum field theory described by the Hamiltonian density in (7) and we rewrite the commutator of \( C = e^0 \mathcal{P}_I \) with \( J^{0i} \) in (27) has two terms:

\[
\begin{align*}
[C, J^{0i}] &= [e^0 \mathcal{P}_I, J_0^{0i} + e J_1^{0i}] \\
&= e^0 [\mathcal{P}_I, J_0^{0i} + e J_1^{0i}] + [e^0, J_0^{0i} + e J_1^{0i}] \mathcal{P}_I \\
&= -2e\mathcal{P}_I J_0^{0i} + [e^0, J_0^{0i} + e J_1^{0i}] \mathcal{P}_I. \quad (29)
\end{align*}
\]

To evaluate the second term in (29) we use the general formula

\[ [f[\pi], \varphi(\mathbf{x}, t)] = -i \frac{\delta}{\delta \pi(\mathbf{x}, t)} f[\pi], \quad (30) \]

from which we obtain

\[
\begin{align*}
&\left[ \exp\left(-\frac{2e}{\mu^2} \int d\mathbf{x} \pi(\mathbf{x}, t)\right), \varphi(\mathbf{x}, t) \right] \\
&= -i \frac{\delta}{\delta \pi(\mathbf{x}, t)} \exp\left(-\frac{2e}{\mu^2} \int d\mathbf{x} \pi(\mathbf{x}, t)\right) \\
&= \frac{2ie}{\mu^2} \exp\left(-\frac{2e}{\mu^2} \int d\mathbf{x} \pi(\mathbf{x}, t)\right). \quad (31)
\end{align*}
\]

Thus, the second term in (29) evaluates to

\[
\begin{align*}
\mathcal{H} &\quad \text{as } \mathcal{H}(\mathbf{x}, t) = \mathcal{H}_0(\mathbf{x}, t) + e \mathcal{H}_1(\mathbf{x}, t), \text{ where} \\
\mathcal{H}_0(\mathbf{x}, t) &= \frac{1}{2} \pi^2(\mathbf{x}, t) + \frac{1}{2} \mu^2 \varphi^2(\mathbf{x}, t) + \frac{1}{2} [\nabla_x \varphi(\mathbf{x}, t)]^2, \\
\mathcal{H}_1(\mathbf{x}, t) &= i\varphi^3(\mathbf{x}, t). \quad (35)
\end{align*}
\]

For this nontrivial field theory the \( C \) operator still has the form \( C = e^0 \mathcal{P}_I \), where \( Q \) is a series in odd powers of \( e \), \( Q = e Q_1 + e^3 Q_3 + e^5 Q_5 + \cdots \), and \( P_I \) is the intrinsic parity operator.

We expect that the operator \( C \) is a rotational scalar because neither \( Q \) nor \( P_I \) depend on spatial coordinates and one can verify that \( C \) is indeed a rotational scalar by showing that \( C \) commutes with the generator of spatial rotations \( J^{0i} \). To prove that \( C \) is a Lorentz scalar, we must show that \( C \) also commutes with the Lorentz boost operator \( J^{0i} = J_0^{0i} + e J_1^{0i} \), where the general formulas for \( J_0^{0i} \) and \( J_1^{0i} \) are given in (28).
Let us expand the commutator $[C, J^{0i}]$ in powers of $\epsilon$. To order $\epsilon^2$, we have

$$
[C, J^{0i}] = \left[ 1 + \epsilon Q_1 + \frac{1}{2} \epsilon^2 Q_1^2 \right] \mathcal{P}_I J^{0i} + \epsilon J^{0i} + O(\epsilon^3)
$$

$$
= [\mathcal{P}_I J^{0i}] + \epsilon Q_1 [\mathcal{P}_I, J^{0i}] + Q_1 [\mathcal{P}_I, J^{0i}] + \epsilon^2 \left( \frac{1}{2} Q_1^2 [\mathcal{P}_I, J^{0i}] + \frac{1}{2} Q_1^2 [\mathcal{P}_I, J^{0i}] \right)
$$

$$
+ [Q_1, J^{0i}] \mathcal{P}_I + Q_1 [\mathcal{P}_I, J^{0i}] + O(\epsilon^3). \quad (36)
$$

The leading term vanishes because $\mathcal{P}_I$ commutes with $J^{0i}$: $[\mathcal{P}_I, J^{0i}] = 0$. Using the identity $[\mathcal{P}_I, J^{0i}] = -2 J^{0i} \mathcal{P}_I$, we simplify (36) to

$$
[C, J^{0i}] = \epsilon Q_1 [\mathcal{P}_I, J^{0i}] - 2 J^{0i} \mathcal{P}_I + \epsilon^2 \left( \frac{1}{2} Q_1^2 [\mathcal{P}_I, J^{0i}] + \frac{1}{2} Q_1^2 [\mathcal{P}_I, J^{0i}] \right)
$$

$$
- Q_1 J^{0i} \mathcal{P}_I - J^{0i} Q_1 \mathcal{P}_I + O(\epsilon^3). \quad (37)
$$

Evidently, if the term of order $\epsilon$ vanishes, then the $\epsilon^2$ term vanishes automatically.

From (37) we can see that to prove that $C$ transforms as a scalar we need to show that $[Q_1, J^{0i}] = 2 J^{0i}$. The generator $J^{0i}$ in (28) consists of two parts, and the first part commutes with $Q_1$. To show this, we argue that for any functional of $\pi$ and $\varphi$, say $f(\pi, \varphi)$, the first term of $J^{0i}$ in (28) commutes with $f(\pi, \varphi)$:

$$
[f(\pi, \varphi), t \int \mathcal{M}(\pi, \varphi)] = 0. \quad (38)
$$

where we have used the variational formulas $[f(\pi, \varphi), \mathcal{M}(\pi, \varphi)] = -i(\delta f(\pi, \varphi)/\delta \mathcal{M}(\pi, \varphi))$ and $[f(\pi, \varphi), \pi(x, t)] = i(\delta f(\pi, \varphi)/\delta \pi(x, t))$ for $f(\pi, \varphi)$. Integrating the second term of (38) by parts, we get

$$
[f(\pi, \varphi), t \int \mathcal{M}(\pi, \varphi)] = t \int dy \left( i \frac{\delta}{\delta \varphi(y, t)} f(\pi, \varphi) \nabla^2 \varphi(y, t) + i \nabla \varphi(y, t) \right)
$$

$$
\times \frac{\delta}{\delta \pi(y, t)} f(\pi, \varphi) = 0. \quad (39)
$$

Thus, since $Q_1$ is a functional of $\pi$ and $\varphi$, only the second term of $J^{0i}$ in (28) contributes to the commutator of $Q_1$ with $J^{0i}$: $[Q_1, J^{0i}] = -[Q_1, \int dx \mathcal{M}(\pi(x, t))].$

Thus, we have reduced the problem of showing that $C$ transforms as a scalar to establishing the commutator identity

$$
[Q_1, \int dx \left( \frac{1}{2} \pi^2(x, t) + \frac{1}{2} \varphi^2(x, t) + \frac{1}{2} \nabla \varphi(x, t)^2 \right)]
$$

$$
= 2i \int dx \varphi^3(x, t). \quad (40)
$$

This equation is similar in structure to Eq. (66) in Ref. [7],

$$
[Q_1, \int dx \left( \frac{1}{2} \pi^2(x, t) + \frac{1}{2} \varphi^2(x, t) + \frac{1}{2} \nabla \varphi(x, t)^2 \right)]
$$

$$
= 2i \int dx \varphi^3(x, t), \quad (41)
$$

apart from an integration by parts. The only difference between (40) and (41) is that there are extra factors of $x^i$ in the integrands of (40).

We will now follow the same line of analysis used in Ref. [7] to solve (40). We introduce the identical ansatz for $Q_1$:

$$
Q_1 = \int dx dy dz M(xyz) \pi x \pi y \pi z
$$

$$
+ \int dx dy dz N(xyz) \varphi x \varphi y \varphi z. \quad (42)
$$

where we have suppressed the time variable $t$ in the fields and have indicated spatial dependences with subscripts. To indicate that the unknown function $M$ is totally symmetric in its three arguments, we use the notation $M_{xyz}$ and we write $N_{xyz}$ because the unknown function $N$ is symmetric under the interchange of the second and third arguments.

Performing the commutator in (40), we obtain two functional equations:

$$
\int dxdydz \varphi x \varphi y \varphi z [(x^i + y^i + z^i) \mu^2 - \nabla x^i - x^i \nabla^2] M_{xyz} = -2 \int dww^3 \varphi_w^3. \quad (43)
$$

$$
\int dxdydz (y^i \pi y \pi x \varphi z + z^i \varphi x \pi y \pi z) N_{xyz}
$$

$$
= \int dxdydz \varphi x \pi y \pi z [(x^i \mu^2 - \nabla x^i - x^i \nabla^2) M_{xyz}
$$

$$
+ \pi y \pi x (y^i \mu^2 - \nabla y^i - y^i \nabla^2) M_{xyz}
$$

$$
+ \pi z \pi x (z^i \mu^2 - \nabla z^i - z^i \nabla^2) M_{xyz}]. \quad (44)
$$

Next, we commute (43) three times with $\pi$ and commute (44) once with $\pi$ and twice with $\varphi$ to transform these operator identities into coupled differential equations for $M$ and $N$:
If these solutions solve (45) and (46) as well, then we can obtain
\begin{align}
\text{LHS of } (\mathcal{N}) = 3 \left[ x^i (\mu^2 - \nabla^2) - \nabla^i \right] M_{\text{xyz}} - 6 \delta(x - y) \delta(x - z). 
\end{align}

Equation (45) is similar to Eq. (71) in Ref. [7]:
\begin{align}
(\mu^2 - \nabla^2) N_{\text{xyz}} + (\mu^2 - \nabla^2) N_{\text{xyz}} + (\mu^2 - \nabla^2) N_{\text{xyz}} 
= -6 \delta(x - y) \delta(x - z). 
\end{align}

By permuting x, y, and z, we rewrite Eq. (72) of Ref. [7] as
\begin{align}
N_{\text{xyz}} + N_{\text{xyz}} = 3(\mu^2 - \nabla^2) M_{\text{xyz}}. 
\end{align}

This equation is similar to Eq. (46). The solutions for M and N are given in Eqs. (83) and (84) of Ref. [7]:
\begin{align}
M_{\text{xyz}} = -\frac{4}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \frac{e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}}{\mathcal{D}(\mathbf{p}, \mathbf{q})}, 
\end{align}

where \( \mathcal{D}(\mathbf{p}, \mathbf{q}) = 4(\mathbf{p}^2 \mathbf{q}^2 - (\mathbf{p} \cdot \mathbf{q})^2) + 4\mu^2(\mathbf{p}^2 + \mathbf{p} \cdot \mathbf{q} + \mathbf{q}^2) + 3\mu^4 \), and
\begin{align}
N_{\text{xyz}} = \frac{12}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \frac{-\mathbf{p} \cdot \mathbf{q} + \frac{1}{2} \mu^2 e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}}{\mathcal{D}(\mathbf{p}, \mathbf{q})}. 
\end{align}

If these solutions solve (45) and (46) as well, then we can claim that the operator C transforms as a scalar. To show that this is indeed true, we multiply (47) by \( x^i \) and then subtract the result from (45) to obtain
\begin{align}
(y^i - x^i)(\mu^2 - \nabla^2) N_{\text{xyz}} + (z^i - x^i)(\mu^2 - \nabla^2) N_{\text{xyz}} 
= \nabla^i N_{\text{xyz}} + \nabla_y N_{\text{xyz}} + \nabla_z N_{\text{xyz}}. 
\end{align}

By making the change of variables \( \mathbf{p} \to \mathbf{q}, \mathbf{q} \to -\mathbf{p}, \mathbf{q} \to \mathbf{p} \), we obtain \( N_{\text{xyz}} \) in the form
\begin{align}
N_{\text{xyz}} = -\frac{12}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \frac{[\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}}{\mathcal{D}(\mathbf{p}, \mathbf{q})}, 
\end{align}

where we have used \( \mathcal{D}(\mathbf{q}, -\mathbf{p} - \mathbf{q}) = \mathcal{D}(\mathbf{p}, \mathbf{q}) \). Similarly, by making the change of variables \( \mathbf{p} \to -\mathbf{p} - \mathbf{q}, \mathbf{q} \to -\mathbf{p} \) and using the identity \( \mathcal{D}(-\mathbf{p} - \mathbf{q}, \mathbf{p}) = \mathcal{D}(\mathbf{p}, \mathbf{q}) \), we obtain
\begin{align}
N_{\text{xyz}} = -\frac{12}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \frac{[\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}}{\mathcal{D}(\mathbf{p}, \mathbf{q})}. 
\end{align}

Thus, from (50), (52), and (53), we find that the right-hand side of (51) becomes
\begin{align}
\text{RHS of } (51) = \frac{12i}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \frac{[\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}}{\mathcal{D}(\mathbf{p}, \mathbf{q})}. 
\end{align}

Also, substituting (52) and (53) into the LHS of (51), we get
\begin{align}
\text{LHS of } (51) = -\frac{12}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \left[ (y^i - x^i)[\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2][\mathbf{p}^2 + \mu^2] + (z^i - x^i)[\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2][\mathbf{q}^2 + \mu^2] \right] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}. 
\end{align}

We must now show that the RHS of (51) in (54) and the LHS of (51) in (55) are equal. To do so, we substitute the identities \( (y^i - x^i)e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}} = i\nabla_y e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}} \) and \( (z^i - x^i)e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}} = i\nabla_z e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}} \) into (55) and obtain
\begin{align}
\text{LHS of } (51) = -\frac{12i}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \left[ \nabla_y \left[ \frac{\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] + \nabla_z \left[ \frac{\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] \right] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}. 
\end{align}

Integration by parts then yields
\begin{align}
\text{LHS of } (51) = \frac{12i}{(2\pi)^6} \int d\mathbf{p} d\mathbf{q} \left[ \nabla_y \left[ \frac{\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] + \nabla_z \left[ \frac{\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] \right] e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}. 
\end{align}
Finally, we substitute the algebraic identity
\[
\nabla_p \left[ \frac{\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] + \nabla_q \left[ \frac{\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] = \frac{\mathbf{p}'(q^2 + 2p \cdot q) + \mathbf{q}'(p^2 + 2p \cdot q)}{\mathcal{D}(\mathbf{p}, \mathbf{q})}
\]

to establish that the LHS and the RHS of (51) are equal. By combining this result with (47), we show that (45) holds.

Using the same technique, we can prove that Eq. (46) holds as well. First, we multiply (48) by \(x'\) and subtract the result from (46) to get
\[
(z' - x')N_{y(xz)} + (y' - x')N_{z(xy)} = -3\nabla_x M_{(xyz)}.
\]

Second, following the technique we used to derive (57), we find that the LHS of (58) becomes
\[
\text{LHS of (58)} = \frac{12i}{(2\pi)^2} \int d\mathbf{p} d\mathbf{q} \left[ \nabla_q \left( \mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2 \right) \mathcal{D}(\mathbf{p}, \mathbf{q}) \right. \\
+ \left. \nabla_p \left( \mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2 \right) \mathcal{D}(\mathbf{p}, \mathbf{q}) \right] \\
\times e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}},
\]

and by substituting (49) into (58), we find that the RHS of (58) becomes
\[
\text{RHS of (58)} = \frac{12i}{(2\pi)^2} \int d\mathbf{p} d\mathbf{q} \left( \mathbf{q}' \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2 \right) \mathcal{D}(\mathbf{p}, \mathbf{q}) e^{i(x-y)\mathbf{p} + i(x-z)\mathbf{q}}.
\]

Finally, we substitute the algebraic identity
\[
\nabla_q \left[ \frac{\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] + \nabla_p \left[ \frac{\mathbf{p} \cdot (\mathbf{p} + \mathbf{q}) + \frac{1}{2} \mu^2}{\mathcal{D}(\mathbf{p}, \mathbf{q})} \right] = \frac{\mathbf{p}' + \mathbf{q}'}{\mathcal{D}(\mathbf{p}, \mathbf{q})}
\]

to establish that the LHS of (58) and the RHS of (58) are equal. Thus, (58) holds and we can combine this result with (48) to establish that (46) is valid. We conclude that to order \(O(\epsilon^2)\) the operator \(C\) transforms as a Lorentz scalar.

\section*{IV. Final Remarks}

We have shown that the operator \(C\) in a toy model \(i\varphi^3\) quantum field theory transforms as a Lorentz scalar and that to order \(O(\epsilon^2)\) the \(C\) operator in an \(i\varphi^3\) quantum field theory transforms as a Lorentz scalar. Based on this work, we conjecture that in general the \(C\) operator for any unbroken \(PT\)-symmetric quantum field theory always transforms as a Lorentz scalar.

By establishing the Lorentz transformation properties of \(C\), we can now begin to understand the role played by this mysterious operator: Apparently, the \(C\) operator is the non-Hermitian \(PT\)-symmetric analog of the intrinsic parity operator \(P\). A conventional Hermitian quantum field theory does not have a \(C\) operator. However, if we begin with the Hermitian theory corresponding to \(\epsilon = 0\) and turn on \(\epsilon\), then the (scalar) \(P\) symmetry of the Hermitian theory disappears and is replaced by the (scalar) \(C\) symmetry of the non-Hermitian theory.

The complex cubic quantum field theory discussed here is especially important because it controls the dynamics of Reggeon field theory \cite{[9]} and describes the Lee-Yang edge singularity \cite{[17]}. The work in this paper shows that an \(i\varphi^3\) field theory is a consistent unitary quantum field theory on a Hilbert space having a \textit{Lorentz invariant} inner product.

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[14] It is worthwhile but very difficult to verify that the \( C \) operator in (10), as opposed to the \( C \) operator in (5), really defines a positive norm for the cubic theory. In the case of the Lee model, the problem with the \( C \) operator in (5) is connected with the appearance of degeneracy: It is possible to have two states having the same energy but with different momentum. In the Lee model, only the \( C \) operator in (10) gives a physical theory having a positive norm and verifying this can be done explicitly (see Ref. [10]).

[15] A way to understand the result in (26) is to observe that adding an imaginary constant to the field \( \varphi \), \( \phi = \varphi + i \epsilon \mu^{-2} \), replaces the Hamiltonian density in (17) by \( \mathcal{H}_{\text{new}}(x, t) = \frac{1}{2} \pi^2(x, t) + \frac{1}{2} \mu^2 \phi^2(x, t) + \frac{1}{4} (\nabla \phi(x, t))^2 + A \), where \( A \) is a real constant. Clearly, \( \mathcal{P}_I \) is not a symmetry of the Hamiltonian associated with \( \mathcal{H}_{\text{new}} \) because \( \mathcal{P}_I \phi \mathcal{P}_I = -\phi + 2i \epsilon \mu^{-2} \). However, \( C \) in (26) is a symmetry of the new Hamiltonian because \( C \phi \mathcal{C} = -\phi \).
