Separation of radial and angular dependence for a central potential

In class we did separation of variables, and obtained

$$\begin{aligned}
&\frac{-\hbar^2}{2m}\frac{r^2}{R}\Big(\frac{\partial^2 R}{\partial r^2}+\frac{2}{r}\frac{\partial R}{\partial r}\Big)+r^2U(r)-Er^2\\
+\;&\frac{-\hbar^2}{2m}\frac{1}{Y}\Big(\frac{\partial^2 Y}{\partial \theta^2}+\cot\theta\frac{\partial Y}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\Big)=0
\end{aligned} \tag{1}$$

This is true for all r,θ,ϕ so the first line (which can depend only on r) and the second line (which can depend only on θ,ϕ) must both be constants, which add to zero. We write the constant as $C\hbar^2/(2m)$, so

$$\begin{aligned}
\frac{-\hbar^2}{2m}\frac{r^2}{R}\Big(\frac{\partial^2 R}{\partial r^2}+\frac{2}{r}\frac{\partial R}{\partial r}\Big)+r^2U(r)-Er^2 &= \frac{C\hbar^2}{2m}\\
\frac{-\hbar^2}{2m}\frac{1}{Y}\Big(\frac{\partial^2 Y}{\partial \theta^2}+\cot\theta\frac{\partial Y}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\Big) &= -\frac{C\hbar^2}{2m}
\end{aligned} \tag{2}$$

Angular equation
Multiply both sides by Y and cancel common factors, obtaining

$$\begin{aligned}
\Big(\frac{\partial^2 Y}{\partial \theta^2}+\cot\theta\frac{\partial Y}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2}\Big) &= CY\\
\text{i.e.}\quad -\frac{1}{\hbar^2}\hat{L}^2 &= CY
\end{aligned} \tag{3}$$

This shows that $Y(\theta,\phi)$ obeys its own eigenvalue equation, which is independent of the potential $U(r)$. This equation simply states that Y is an eigenstate of the total orbital angular momentum operator. The equation can only be solved when $C=-l(l+1)$, for integer $l\geqslant 0$.

Radial equation
Multiply both sides by R/r^2, obtaining

$$\frac{-\hbar^2}{2m}\Big(\frac{\partial^2 R}{\partial r^2}+\frac{2}{r}\frac{\partial R}{\partial r}+\frac{C}{r^2}R\Big)+U(r)R = ER \tag{4}$$

This shows that $R(r)$ obeys an eigenvalue equation similar to the ones we found in one-dimensional systems, except that there is an additional single-derivative term in the kinetic energy, and an additional repulsive contribution $-\hbar^2C/(2mr^2)$ to the potential. (It is repulsive because C is negative, see above). C reflects the amount of angular momentum, and this term is called the "centrifugal barrier".