Physics 217
Problem Set 1
Due: Friday, Aug 29th, 2008

Course page: `http://www.physics.wustl.edu/~alford/p217/`

Review of complex numbers. See appendix K of the textbook.

1. Consider complex numbers $z = 1.5 + 0.5i$ and $w = -0.6 + 0.3i$, where $i^2 = -1$.

   (a) Write z and w in the modulus-argument form $re^{i\theta}$, in each case giving the numerical values of the modulus r, and the argument θ (in radians).

   (b) Write down the complex conjugates z^* and w^*.

   (c) Calculate $|z|^2$ and $|w|^2$.

   (d) Calculate $z\,w$ (give numerical answer in modulus-argument form).

   (e) Calculate z/w (in modulus-argument form again).

   (f) Calculate z^*w (in modulus-argument form).

   {10}

2. Consider a time-dependent complex-valued function, $f(t) = e^{i\omega t}$.

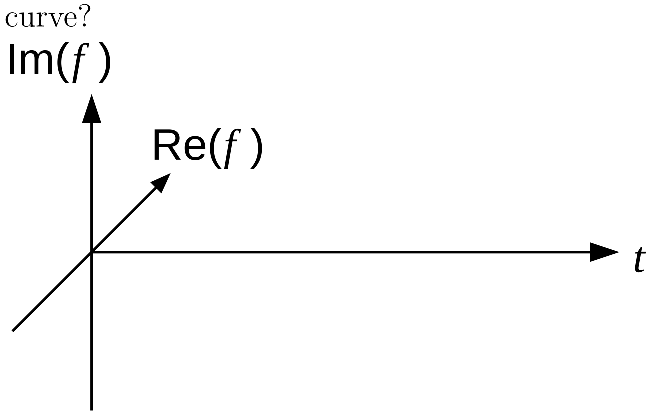
   (a) Draw the shape traced out in the complex plane by the value of $f(t)$ as t varies.

   (b) At time $t = 0$ the value of the function is 1; how long do you have to wait before the function has the same value again?

   (c) Is $f(t)$ a periodic function? If so, what is its period?

   (d) Sketch a graph of how the real part of f (written Re(f)) varies with time. On the same plot, add another curve showing how the imaginary part of f (written Im(f)) varies with t.

   (e) On a 3D plot with the axes as shown below, draw a single curve showing how $f(t)$ moves in the complex plane as a function of t. What is the name of this curve?



   {10}

Physics 217
Problem Set 2
Due: Friday, September 5th, 2008

1. Sunlight has a typical wavelength of 500 nm. What is the energy (in eV) of a photon of that wavelength? Can sunlight produce photoelectrons from metals, given that most metals have work functions in the range 2 to 5 eV? **{5}**

2. Electromagnetic radiation of an unknown wavelength shines on a clean surface of Magnesium, which has a work function of 3.7 eV. Electrons of energy 1.6 eV are emitted. What is the wavelength of the light? In which part of the electromagnetic spectrum is the radiation? **{5}**

3. Harris, Chapter 3, problem 36. **{10}**

4. Harris, Chapter 3, problem 50. **{10}**

Physics 217
Problem Set 3
Due: Friday, September 12th, 2008

1. In an experiment of the type shown in Fig. 4.6, X-rays of wavelength $\lambda = 0.18$ nm are diffracted off a crystal, and maxima in their intensity are observed at $\theta = 11.6°$ and $\theta = 23.9^0$ What is the spacing d of the atomic planes? [compare Harris, Ch 4, problem 13]. **{5}**

2. At what speed would an electron have a de Broglie wavelength of 5 μm? [compare Harris, Ch 4, problem 15] **{5}**

3. Harris Ch 4, problem 22 **{10}**

4. You can treat a baseball quantum-mechanically. What is the de Broglie wavelength of a fast-pitched baseball? (The mass of a baseball is 0.145 kg, and the speed of a fast pitch is about 40 m/s.) Now, imagine we have pointlike baseballs: they will make an interference pattern when passed through two slits. Suppose the two slits are 50 cm apart. What is the angular separation between successive maxima in the interference pattern? **{10}**

Physics 217
Problem Set 4
Due: Friday, September 19th, 2008

1. A particle in one dimension has the wavefunction

$$\psi(x) = \begin{cases} A\cos(\pi x/a), & -a/2 < x < +a/2 \\ 0 & x < -a/2 \text{ or } x > +a/2 \end{cases}$$

   (a) Sketch a graph showing $\psi(x)$.
   (b) Determine the constant A by normalizing the wavefunction.
   (c) Calculate the expected value of position x, $\langle\psi|\hat{x}|\psi\rangle$.
   (d) Calculate the expected value of x^2, $\langle\psi|\hat{x}^2|\psi\rangle$.
   (e) Calculate the expected value of momentum p, $\langle\psi|\hat{p}|\psi\rangle$.
   (f) Calculate the expected value of p^2, $\langle\psi|\hat{p}^2|\psi\rangle$.
   (g) Calculate the uncertainty in position, Δx.
   (h) Calculate the uncertainty in momentum, Δp.
   (i) Verify that the Heisenberg uncertainty relation $\Delta x\,\Delta p \geqslant \hbar/2$ is obeyed.

   {20}

2. In this question we use the Heisenberg uncertainty relation, $\Delta p\,\Delta x \geqslant \hbar/2$, to estimate the ground state energy of a particle in a harmonic oscillator potential, for which the energy is

$$E = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2$$

   In the classical limit ($\hbar \to 0$) the particle could have $x = 0$ and $p = 0$, so the smallest energy it could have would be zero.

   (a) In quantum mechanics, the particle has some uncertainty Δx in its position: make a rough estimate of the resultant contribution to its potential energy.
   (b) For a given Δx, obtain a lower limit on the uncertainty in its momentum. Assuming that the momentum can be near its lower limit, estimate the resultant contribution to the particle's kinetic energy.
   (c) Now find the value of Δx for which the total energy is the smallest. What do you find is the minimum possible energy the particle can have?

   {10}

Physics 217
Problem Set 5
Due: Friday, September 26th, 2008

1. Harris, Ch 5, problem 24. **{5}**

2. Harris, Ch 5, problem 28. **{10}**

3. Consider an electron in an infinitely deep potential well of length $L = 2$ nm. The normalized eigenstates of the Hamiltonian are $\psi_n(x)$ $(n = 1, 2, \ldots)$ given by Harris (5-16). Suppose that at time $t = 0$ the wavefunction of the system is an equal superposition of the first two eigenstates,

$$\psi(x, 0) = \frac{1}{\sqrt{2}}\Big(\psi_1(x) + \psi_2(x)\Big)$$

In this question we study the time-dependence of this state.

(a) What are the energies E_1 and E_2 of the first two energy eigenstates ψ_1 and ψ_2? (in Joules or eV)

(b) Write equations showing how ψ_1 and ψ_2 vary in time. Write them in terms of E_1, E_2, and L: don't plug in their values yet. (Hint: see sect (5.2).)

(c) What is $\psi(x, t)$, for arbitrary time t? (Again, don't plug in numbers yet).

(d) Draw a sketch of $|\psi(x, 0)|^2$, the probability distribution at time $t = 0$. (Now you can plug in numbers if you want to.)

(e) Draw a sketch of $|\psi(x, t_1)|^2$, where $t_1 = 7 \times 10^{-15}$ s.

(f) If you had to roughly describe the behavior of the wavefunction in classical terms, what would you say the particle is doing?

(g) Obtain an expression for the probability distribution as a function of time, $P(x, t) = |\psi(x, t)|^2$, in terms of E_1, E_2, and L. Show that it is periodic in time. What is its period?

{20}

Physics 217
Problem Set 6
Due: Friday, October 3rd, 2008

1. Show that the eigenstates of the energy in an infinite potential well (Harris (5-16)) are *orthogonal functions*, i.e. that

$$\int_{-\infty}^{+\infty} \psi_m^*(x)\psi_n(x)dx = \delta_{m,n}$$

where m and n are positive integers that label the eigenstates (i.e. m is *not* the mass!) and the "Kroenecker delta" $\delta_{m,n}$ is 1 if $m = n$ and zero otherwise.
For the case $m \neq n$ you can use the fact that

$$\int_0^{\pi} \sin(nz)\sin(mz)\,dz = \frac{n\cos(n\pi)\sin(m\pi) - m\cos(m\pi)\sin(n\pi)}{m^2 - n^2}\ .$$

{10}

2. Suppose that a particle in an infinite well is in a state φ that is a general superposition of the first two energy eigenstates,

$$\varphi(x) = A\psi_1(x) + B\psi_2(x)$$

where A and B are complex coefficients. (This is a generalization of the state we considered in the last homework, where we chose $A = B = 1/\sqrt{2}$.)

   (a) Using the results of the previous question, write down the condition that A and B must obey to ensure that φ is a normalized wavefunction.

   (b) Using the fact that energy eigenstates obey $\hat{H}\psi_n = E_n\psi_n$, and the results of the previous question, calculate the expectation value of the energy in the state φ, i.e. $\langle\varphi|\hat{H}|\varphi\rangle$, in terms of A, B, E_1, and E_2.

   (c) Using the results of the previous two parts, what would you guess is the probability $P(E_1)$ that if someone measures the energy of the particle in state φ, they will find it is E_1? What is $P(E_2)$? What is $P(E_3)$?

{10}

3. In class we derived the energy eigenvalue equation for a particle bound in a square well of width L and depth V_B. We only treated the even solutions, for which $\psi = A\cos(qx)$ inside the well and $\psi = C\exp(-\kappa|x|)$ outside the well. We matched these at $x = a$, and obtained

$$q\tan(qL/2) = \kappa(q)$$

$$\text{where} \quad \kappa(q) = \sqrt{\frac{2mV_B}{\hbar^2} - q^2}\ .$$

Derive the matching equation for the *odd* solutions, for which $\psi = A\sin(qx)$ inside the well. Show that if the well is very shallow, there are no odd bound states. **{10}**

Physics 217
Problem Set 7
Due: Friday, October 10th, 2008

1. Harris, Ch 4, problem 58. **{10}**

2. Consider a particle in the ground state of an infinite well of width L, so its normalized wavefunction is

$$\begin{aligned}\psi(x) &= \sqrt{\tfrac{2}{L}}\sin(\pi x/L) \quad (0 < x < L) \\ &= 0 \quad\quad\quad\quad\quad\quad\; (x < 0,\ x > L)\end{aligned}$$

   (a) Calculate $\tilde{\psi}(k)$ (which is written as "$A(k)$" in the textbook), the Fourier transform of $\psi(x)$.

   (b) Calculate $|\tilde{\psi}(k)|^2$ and draw a sketch of it.

   (c) Since $\psi(x)$ is normalized, $\tilde{\psi}(k)$ should be normalized too. For the definition of the Fourier transform that we are using, it should obey

$$\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = \frac{1}{2\pi} \ .$$

   Check that $\tilde{\psi}(k)$ has this normalization. (If you can't do the integral analytically, do it numerically, or at least show that it is just a constant, independent of L.)

   (d) Make a rough estimate of the uncertainty Δx in the position of the particle. Make a rough estimate of the uncertainty Δk in the wavenumber of the particle. (Δk is the standard deviation of the $|\tilde{\psi}(k)|^2$ distribution, which is roughly the width of the central maximum.) Write down $\Delta x \Delta k$ and hence $\Delta x \Delta p$. Is your result at least roughly consistent with Heisenberg's uncertainty relation?

{15}

Physics 217
Problem Set 8
Due: Fri, October 24th, 2008

1. Harris, Ch 4, problem 62.
   The question should really say "What is the *approximate* range of frequencies..."
   {10}

2. Harris, Ch 4, problem 72. **{10}**

3. Consider a wavefunction defined by

$$A(k) = \delta(k - k_1) + \delta(k + k_1)$$

   where δ is the Dirac delta function (see "Essentials of Quantum Mechanics" notes).

   (a) Is this a normalizeable wavefunction?

   (b) If one measured the momentum of the particle, what are the possible values that could be obtained? What are their relative probabilities?

   (c) Write down the position space wavefunction $\psi(x)$ of the particle.

   (d) If one measured the position of the particle, what are the possible values that could be obtained?

   {10}

Physics 217
Problem Set 9
Due: Friday, Oct 31st, 2008

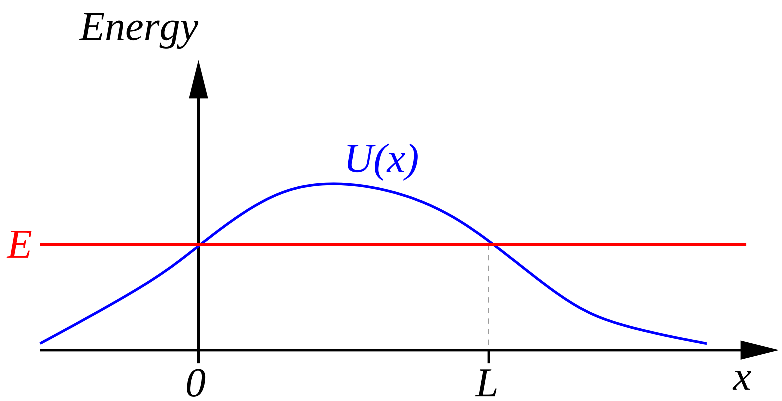
1. An electron is travelling to the right from $x = -\infty$. In the region $x < 0$ it feels no forces, and has a kinetic energy of 5 eV. At $x = 0$ it encounters a *downward* potential step, where the potential energy drops by 4 eV (and then remains constant for $x > 0$). What is the probability that the electron will be reflected back off the step? (Compare Harris, Ch 6, problem 15).
   {10}

2. Harris, Ch 6, problem 24. **{10}**

3. Solve the matching equations for the wavefunction of a particle of energy E striking a potential barrier of height U_0 and length L, where $E > U_0$. Obtain expressions for the reflection amplitude B and transmission amplitude F in terms of the incoming amplitude A. Show that your results are consistent with the expressions for the reflection and transmission probabilities R and T given in the textbook, equation (6.12).
   {10}

Physics 217
Problem Set 10
Due: Friday, Nov 7th, 2008

1. Harris, Ch 6, problem 37.
   A sketch of the situation:



   The behavior of the potential in the regions $x < 0$ and $x > L$, where $E > U(x)$, is not important. In the expression for T, the limits on the integral should be 0 to L (not 1 to 2).

   {10}

2. Using the result of the previous question, calculate the tunnelling probability T for an electron of mass m with very small energy ($E \approx 0$) to escape from a metal with work function W in electric field $\mathcal{E}$. This can be regarded as a potential barrier problem: the region $x < 0$ is the metal, where $U(x) = 0$; the region $x > 0$ is outside the metal, where $U(x) = W - e\mathcal{E}x$. **{10}**

3. Harris, Ch 6, problem 46.
   Make a sketch of $\omega(k)$ as well as of the group velocity. **{10}**

Physics 217
Problem Set 11
Due: Friday, Nov 14th, 2008

1. Read Harris, Ch 6, problem 49. Harris gives an expression for Δx, which is the size of a Gaussian wavepacket at time t. The wavepacket started at time $t = 0$ with size ϵ. The rest of this question concerns Gaussian wavepackets of this type.

   (a) Sketch how the size of the wavepacket varies in time, i.e. sketch $\Delta x(t)$ for fixed ϵ. How does the function behave at very small t and at very large t?

   (b) Calculate how long it takes the uncertainty in the position of the particle (i.e the size of the wavepacket) to rise to twice its initial value, and obtain the numerical value of this time for an electron wavepacket with initial size $\epsilon = 1$ cm.

   (c) Sketch how the size of a wave packet after a fixed time t depends on its initial size, i.e. sketch $\Delta x(\epsilon)$ for fixed t. How does the function behave at very small ϵ and very large ϵ?

   (d) What is the minimum size a wavepacket can have after time t?

   (e) An electron wavepacket has size $\Delta x = 3$ cm after it has evolved for 1 second. What could its initial size have been? Explain why there is more than one possible answer.

   (f) Explain why it is impossible for an electron wavepacket to be smaller than 1 cm in size after it has evolved for 1 second.

   {20}

2. Harris, Ch 7, problem 22. **{10}**

3. Harris, Ch 7, problem 28. **{5}**

Physics 217
Problem Set 12
Due: Friday, Nov 21, 2008

1. Harris, Ch 7, problem 32. **{5}**

2. (a) The spherical harmonics $Y_{l,m_l}(\theta,\phi)$ are eigenstates of the angular part of the ∇^2 operator, obeying $r^2\nabla^2 Y_{l,m_l} = -l(l+1)Y_{l,m_l}$, which in spherical co-ordinates (and using the fact that the Y_{l,m_l} are independent of r) is

$$\frac{d^2Y_{l,m_l}}{d\theta^2} + \cot\theta\,\frac{dY_{lm_l}}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2Y_{l,m_l}}{d\phi^2} = -l(l+1)Y_{l,m_l}\ .$$

By writing the spherical harmonics as a product $Y_{l,m_l}(\theta,\phi) = \Theta_{l,m_l}(\theta)\Phi_{m_l}(\phi)$, show that this is equivalent to Harris eqns (7-21) and (7-22).

(b) Show that the $l = 1, m_l = +1$ spherical harmonic (from table 7.3 on p 257) is a solution of the equation written above.

(c) Show that if the spherical harmonic function is to be single-valued, m_l must be an integer.

{15}

3. Consider a free particle in three dimensions.

(a) Free particle in Cartesian co-ordinates. Show that the plane wave

$$\psi_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})$$

is an eigenstate of the Hamiltonian (for any $\vec{k}$). What is its energy?

(b) Free particle in spherical co-ordinates. Show that the spherical wave

$$\psi_{E,l,m_l}(r,\theta,\phi) = R_{E,l}(r)Y_{l,m_l}(\theta,\phi)$$

is an eigenstate of the Hamiltonian with energy E, if $R_l(r)$ obeys the appropriate differential equation. Write down that equation. You can use the properties of the spherical harmonics that are stated in question 2.

(c) Explain how it is possible that the eigenstates of the free particle Hamiltonian are plane waves (according to part (a)) and are spherical waves (according to part (b)). Is there any contradiction here? What do you think is the relationship between the plane waves with a given energy, and the spherical waves with the same energy?

{10}

Physics 217
Problem Set 13
Due: Friday, Dec 5th, 2008

Please remember to supply your evaluation of the course at the course evaluation website, `http://evals.wustl.edu`.

1. Harris, Ch 7, problem 54. By "most probable radius" he means the radius at which the radial probability density has its maximum value. **{10}**

2. Harris, Ch 7, problem 58. **{10}**

3. This is a problem about the Zeeman splitting, so you should ignore the effects of the electron spin (even though they can't be ignored in reality).

   (a) Draw an energy-level diagram showing the $n = 3$ and $n = 2$ energy levels of a Hydrogen atom. How many allowed values of l are there for each?

   (b) Now consider a Hydrogen atom in a magnetic field. Show in an energy level diagram how the $n = 3, l = 2$ level and the $n = 2, l = 1$ level each split up. How many levels does each split into?

   (c) Show on your diagram all the possible transitions between the $n = 3, l = 2$ and $n = 2, l = 1$ levels. How many are there?

   (d) Now assume that only transitions that change m by $0, \pm 1$ are allowed. Draw a diagram showing those. How many are there? How many different frequencies of light will be emitted?

   (e) If the magnetic field is $B = 1$ T, what will those frequencies be?

   {15}