

Physics 318
Problem Set 1
Due: Friday, Jan 27th, 2006

1. The spin-up and spin-down states of an electron can be written in 2-component vector notation as $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. As discussed in class, the spin operators are then 2×2 matrices, $\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- (a) Show that the two states χ_+ and χ_- are orthonormal.
 (b) Show that χ_+ and χ_- are eigenstates of the \hat{S}_z operator. What are their eigenvalues? Show that they are not eigenstates of \hat{S}_x or \hat{S}_y .
 (c) In analogy to the normalization condition for continuum wavefunctions $\int \psi(x)^* \psi(x) dx = 1$, the normalization condition for a spin wavefunction φ is $\varphi^\dagger \cdot \varphi = 1$, where φ^\dagger is the “adjoint” of φ , ie the complex conjugate of its transpose.

Is the state $\phi = \begin{pmatrix} \frac{\sqrt{3}}{2} e^{-2i} \\ \frac{1}{2} \end{pmatrix}$ normalized? What is the normalization

condition for a general state $\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$, where φ_\pm are complex numbers?

- (d) If you measure S_z for an electron in a normalized spin state φ , the probability to find spin up is $P_+ = |\varphi_+|^2$, and the probability to find spin down is $P_- = |\varphi_-|^2$. Show that this is equivalent to writing $P_+ = |\chi_+^\dagger \cdot \varphi|^2$ and $P_- = |\chi_-^\dagger \cdot \varphi|^2$. Write down a state with $P_+ = \frac{2}{3}, P_- = \frac{1}{3}$.
 (e) In analogy to the expression $\langle \hat{A} \rangle = \int \psi(x)^* \hat{A} \psi(x) dx$, the expectation value of a spin operator \hat{A} in a normalized state φ is $\varphi^\dagger \cdot \hat{A} \cdot \varphi$. What is the expectation value of the z -component of the spin in the state $\phi = \begin{pmatrix} \frac{\sqrt{3}}{2} e^{-2i} \\ \frac{1}{2} \end{pmatrix}$?

- (f) What is the expectation value of S_z in a general state $\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$?

2. A first exercise with the “ket” notation. The spin-up and spin-down states of an electron can be written as kets $|\chi_+\rangle$ and $|\chi_-\rangle$. You can think of this as a new notation for column vectors. The adjoints of these are the row vectors (“bra”s) $\langle \chi_+|$ and $\langle \chi_-|$. The orthonormality of these is written in terms of the dot products:

$$\langle \chi_+ | \chi_+ \rangle = 1, \quad \langle \chi_- | \chi_- \rangle = 1, \quad \langle \chi_+ | \chi_- \rangle = \langle \chi_- | \chi_+ \rangle = 0. \quad (\text{ON})$$

A general state is a ket $|\varphi\rangle = \varphi_+ |\chi_+\rangle + \varphi_- |\chi_-\rangle$, and the adjoint of this is the corresponding bra $\langle \varphi| = \varphi_+^* \langle \chi_+| + \varphi_-^* \langle \chi_-|$.

- (a) Use the orthonormality conditions (Eq. (ON) above) to express the normalization condition $\langle\varphi|\varphi\rangle = 1$ in terms of φ_+ and φ_- . Does this agree with the result of question 1 in this homework?
- (b) Show that the probability that an electron in state $|\varphi\rangle$ is spin up, $P_+ = |\langle\chi_+|\varphi\rangle|^2$, agrees with the result of question 1 in this homework.
- (c) Show that if we write the S_z operator as

$$\hat{S}_z = \hbar/2\left(|\chi_+\rangle\langle\chi_+| - |\chi_-\rangle\langle\chi_-|\right)$$

then the expectation value of S_z in a general state $|\varphi\rangle$, ie $\langle\varphi|\hat{S}_z|\varphi\rangle$, agrees with the result of question 1 in this homework.

- 3. (Rephrasing of Bernstein, Fishbane, and Gasiorowicz, Chapter 9, problem 22).
Consider the $n = 3$, $l = 2$ states of singly-ionized Helium, i.e. a single electron bound to a $Z = 2$ nucleus. In zero magnetic field, these states all have the same energy. Now suppose a magnetic field of magnitude 3.2 T is applied to the system. The energy level now splits apart into multiple levels because the different states have different L_z (Zeeman effect) and S_z (z -component of spin). What is the energy difference between the highest and the lowest of these levels?
- 4. Bernstein, Fishbane, and Gasiorowicz: Chapter 9, problem 26.
- 5. Bernstein, Fishbane, and Gasiorowicz: Chapter 9, problem 28.

Physics 318
Problem Set 2
Due: Friday, Feb 6th, 2006

1. This problem concerns the $n = 2$ states of a Hydrogen atom in zero external magnetic field.
 - (a) Ignoring electron and proton spin, what is the energy and degeneracy of the $n = 2$ state?
 - (b) If we include electron and proton spin degrees of freedom, but ignore their effects on the Hamiltonian, what is the degeneracy?
 - (c) Now take into account the effect of the spins in the Hamiltonian, i.e. spin-orbit coupling (fine structure) and hyperfine structure. Calculate how the $n = 2$ levels split up. Draw a figure showing the new set of levels. Give the energy (relative to the state in part (a)), quantum numbers and degeneracy of each level. Check that the total number of states is the same as in part (b).

For the spin-orbit coupling, you need $\langle 1/r^3 \rangle$ in the $n = 2, l = 1$ state. Use the appropriate radial wavefn from Table 9-2 (p245) and the fact that

$$\int_0^{\infty} x^n \exp(-x) dx = \Gamma(1 + n) = n! .$$

(See also example 9-6.)

2. Suppose we had a Hydrogen atom made of a proton and an “illectron”. The illectron is the same mass and charge as an electron, but it has spin quantum number $s = 3/2$ instead of $s = 1/2$. Consider an atom where the illectron is in a p -state ($l = 1$) with some energy E .
 - (a) What is the squared spin S^2 of the illectron? What is the squared orbital angular momentum L^2 ?
 - (b) What are the possible values of the quantum number for total angular momentum of the atom, j ?
 - (c) What are the possible values of the squared total angular momentum J^2 ?
 - (d) Suppose we want to calculate the spin-orbit coupling contribution to the Hamiltonian. What are the possible values of $\vec{L} \cdot \vec{S}$? Once we take spin-orbit coupling into account, our original p -state energy level splits up: how many different levels do we end up with? How many of them rise above the original level E and how many of them drop below E ?

Physics 318
Problem Set 3
Due: Friday, Feb 10th, 2006

1. Consider a potential barrier of height V_0 that peaks at $x = 0$. Near $x = 0$ such a barrier generically has the form

$$V(x) = V_0(1 - x^2/a^2) + \text{higher order in } x/a .$$

Suppose a particle of mass m approaches the barrier from the left with energy E , such that it is almost able to get over the barrier:

$$E = V_0 - \mathcal{E}, \quad (\mathcal{E} \ll V_0)$$

- (a) At what value of x does the particle begin to tunnel?
- (b) At what value of x does the tunnelling particle emerge on the other side of the barrier?
- (c) Write down an approximate expression, in terms of an integral, for the transmission coefficient $|T|^2$, i.e. the probability for the particle to pass through the barrier.
- (d) Evaluate the integral. You may use the fact that

$$\int_{-1}^1 \sqrt{1 - u^2} du = \pi/2 .$$

Check that your answer is dimensionally correct.

- (e) For α -decay of a nucleus, the particle is an α -particle with mass $mc^2 \approx 4000$ MeV; a typical barrier height is $V_0 \approx 10$ MeV, and a typical barrier thickness is $a \approx 5$ fm. Using $\hbar c = 197$ MeV fm, calculate (in MeV) what energy $E = V_0 - \mathcal{E}$ is required to get a transmission coefficient $|T|^2 \sim 0.01$.
 - (f) Do a self-consistency check: the expansion of the potential in powers of x/a will be valid as long as $x^2/a^2 \ll 1$ during the tunnelling. Is that the case in the α -particle example?
 - (g) Use the Boltzmann constant to find out what temperature your value of the energy E corresponds to.
2. Consider a two-state system (it could be two bound states of some molecule, for example) with the hamiltonian $\hat{H} = -E_0 \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}$ where E_0 and δ are both real and positive, and $\delta < 1$
- (a) Calculate the eigenstates ϕ_1 and ϕ_2 of the hamiltonian, and normalize them. What are their eigenvalues?
 - (b) Are the eigenstates orthogonal to each other?
 - (c) Which eigenstate has the lower energy?
 - (d) Suppose $E_0 = 1$ eV and $\delta = 0.01$; what is the wavelength of a photon emitted in a transition between the two eigenstates?

Physics 318
Problem Set 4
Due: Friday, Feb 17th, 2006

These questions are about multi-particle states in the infinite square well. The single-particle infinite square well states are often written $u_1(x)$, $u_2(x)$, etc. However, we label the particles by integers 1, 2, 3... so it can be confusing to label the states by integers too. I suggest that you label the states with letters such as u_g , u_h , etc. So “particle 2 in state u_n ” corresponds to “ $u_n(x_2)$ ”. Remember, for full credit you should give some justification for your answers, including the normalization factors for the wavefunctions.

Also, remember that a spin-0 particle only has a space wavefunction, it does not have a spinor because it has no spin ($s = 0$ so $m_s = 0$).

1. Consider two spin-0 particles in an infinite well. The normalized single-particle energy eigenstates are $u_n(x)$, for $n = 1$ to infinity. Write down the most general two-particle state that is symmetric under exchange of the two particles, including the overall normalization factor (Hint: there are two cases to be treated). Write down the most general antisymmetric two-particle state, including its overall normalization factor.
2. Consider three identical spin-0 bosons in an infinite square well potential. Give the normalized wavefunction of the three-particle ground state, and the three-particle first excited state.
3. Consider two electrons in an infinite square well potential, one in the single-particle ground state $u_g(x)$ and the other in the single-particle first excited state $u_h(x)$.
 - (a) Write down the normalized two-particle wavefunction in the case where the total spin quantum number $s^{\text{tot}} = 0$.
 - (b) Write down the normalized two-particle wavefunction in the case where the total spin quantum number $s^{\text{tot}} = 1$ and the z -component quantum number $m^{\text{tot}} = +1$.
 - (c) Write down the normalized two-particle wavefunction in the case where $s^{\text{tot}} = 1$ and $m^{\text{tot}} = 0$.

Physics 318
Problem Set 5
Due: Friday, Feb 24th, 2006

1. Bernstein, Fishbane, and Gasiorowicz: Chapter 10 problem 16.
2. Electrons in a three-dimensional box have Fermi momentum p_F .
 - (a) What is the density of electrons?
 - (b) What is the Fermi energy E_F ?
 - (c) Suppose I randomly select an electron in the box, and measure its momentum p and kinetic energy E . Calculate the following probabilities.
 - (i) $\text{Prob}(p > p_F)$
 - (ii) $\text{Prob}(p > p_F/2)$
 - (iii) $\text{Prob}(E > E_F)$
 - (iv) $\text{Prob}(E > E_F/2)$

Physics 318
Problem Set 6
Due: Friday, Mar 3, 2006

1. Use equations (10-67) and (10-68) to derive an expression for the radius of a stable white dwarf star, in which electron degeneracy pressure balances the pressure due to gravity. (The answer is given as equation (10-69), but that equation is not correct as printed in the book.)
2. The “bulk modulus” B of a material is a measure of how hard it is to compress. The definition is

$$B = -V \left. \frac{dp}{dV} \right|_N$$

where $p(V)$ is the pressure as a function of the volume. Since compressing a sample changes the volume but does not change the number of particles N , the derivative is at fixed N .

Calculate the bulk modulus of Copper, treating Copper as a degenerate gas of electrons. The steps are:

- (a) Using results derived in class, write down the electron degeneracy pressure in terms of the electron density.
- (b) Rewrite the electron density $n_e = N_e/V$ and take the derivative with respect to V at fixed N_e to obtain B .
- (c) Express B in terms of n_e .
- (d) Obtain n_e from the atomic mass of Copper $A = 63.5$, its density $\rho = 8950 \text{ kg/m}^3$, and its valence $\eta = 1$. Calculate the numerical value of B for Copper. Compare your result with the measured value, $1.34 \times 10^{13} \text{ N/m}^2$. Why are they so different?

Physics 318
Problem Set 7
Due: Friday, Mar 24th, 2006

1. How many valence (i.e. unpaired) electrons does Boron have? What is their total spin quantum number and their z -component-of-spin quantum number? Write down their spin wavefunction. What are the possible values for their total orbital angular momentum quantum number?
2. How many valence electrons does Nitrogen have? What is their total spin quantum number s^{tot} ? What are the possible values of m_s^{tot} , their z -component-of-spin quantum number? Write down their spin wavefunction for the case where m_s^{tot} has its maximum value. What are the possible values of the total orbital angular momentum quantum number l^{tot} for the valence electrons? Remember that the full spin+space wavefunction must be antisymmetric.
3. Explain the following features of table 11-2:
 - (a) Atoms can have electron configurations that include “ $3p$ ” (which means $3p^1$) up to $3p^6$. Why is there no $3p^7$?
 - (b) What is the meaning of the term “ionization energy”? Why is the ionization energy of Potassium much less than that of Argon?
 - (c) Which is the largest atom in the table? What is its radius in nanometers? Why is it large relative to the others?
 - (d) Which is the smallest atom? What is its radius in nanometers?
 - (e) Why do the radii of atoms vary over a relatively small range, given that the number of electrons in the atom is varying from 1 to 42?
 - (f) Describe and explain the relationship between ionization energy and radius for atoms.

Physics 318
Problem Set 8
Due: Friday, March 31, 2006

1. An X-ray can knock an inner-shell electron out of an atom. What is the longest wavelength of photon that can knock an electron out of the $1s$ shell of Manganese ($Z = 25$)?
2. Bernstein, Fishbane, and Gasiorowicz: Chapter 11 problem 6.
3. The mass of a proton is $938.271998(4)$ MeV, the mass of an electron is $0.510998902(2)$ MeV. (The number in parentheses is the measurement uncertainty in the last digit of the mass). What is the mass of a Hydrogen atom in its ground state? Give the most accurate possible answer, and include the uncertainty.

Physics 318
Problem Set 9
Due: Friday, April 7, 2006

1. The vibrational energy spectrum of a molecule consists of equally-spaced levels, with spacing $\Delta E_{\text{vib}} = \hbar\omega$.
 - (a) What is the frequency of a photon emitted in a transition from one vibrational level to the next-lowest level?
 - (b) What is the wavelength λ of such a photon? This is the “characteristic wavelength” of the vibrational spectrum.
 - (c) Chlorine comes in two isotopes, ^{35}Cl and ^{37}Cl , with atomic weights $A = 35$ and $A = 37$ respectively. If we know that for the molecule H^{35}Cl the inverse characteristic wavelength of the vibrational spectrum is $1/\lambda = 2990 \text{ cm}^{-1}$, what is the value of $1/\lambda$ for the molecule H^{37}Cl ?

2. Bernstein, Fishbane, and Gasiorowicz: Chapter 11 problem 12.

3. Bernstein, Fishbane, and Gasiorowicz: Chapter 11 problem 18.

Physics 318
Problem Set 10
Due: Friday, April 14, 2006

1. Categorize the following as bosons or fermions. Justify your answers.
 - (a) Hydrogen atom.
 - (b) Deuterium atom (nucleus is one neutron and one proton).
 - (c) π^+ ($u\bar{d}$).
 - (d) Δ^{++} (uuu).

2. Write down the lowest-order Feynman diagram for each of the following processes.
 - (a) An electron is scattered by a positron.
 - (b) An electron and a positron annihilate to two photons.
 - (c) A photon scatters off an electron (Compton scattering).
 - (d) A photon scatters off another photon. (Hint: of course, photons do not interact directly, but suppose one of them temporarily turned into a virtual electron-positron pair. . .).

3. Suppose the photon had a very tiny mass $m_\gamma = 10^{-5}$ eV. At what distance would the electrostatic potential of a charge show a 10% difference from the Coulomb law? Would the discrepancy from Coulomb's law (as a percentage) be greater at shorter distances or at longer distances?

Physics 318
Problem Set 11
Due: Friday, April 21, 2006

1. Which conservation laws would be violated by each of the following (never observed) processes?
 - (a) $e^- \rightarrow \nu_e \gamma$
 - (b) $p \rightarrow \pi^+ \gamma$
 - (c) $\Lambda^0 \rightarrow p K^-$
 - (d) $\pi^+ \rightarrow e^+ e^+ e^-$

2. In scattering experiments, there is not enough time for the weak interaction to have a significant effect, so all the processes that occur are mediated by the strong interaction, which conserves flavor. Draw quark flow diagrams for the following processes:
 - (a) $\pi^- p \rightarrow K^0 \Lambda^0$
 - (b) $p n \rightarrow p n \bar{p} p$
 - (c) $K^- p \rightarrow \Sigma^0 \pi^0$
 - (d) $K^- p \rightarrow \Xi^- K^+$

3. In weak interactions, the flavor of the quarks can change. An up-type quark (u, c, t) can emit a W^+ and turn into a down-type quark (d, s, b) or vice versa. The Ξ^- particle has been observed to decay like this: $\Xi^- \rightarrow \Lambda^0 \pi^-$. Draw a quark flow diagram, including the virtual W^\pm boson, for this decay. The alternative decay mode $\Xi^- \rightarrow n \pi^-$, is possible, but has not yet been observed. Draw a possible quark flow diagram for this alternative decay, and explain why it is so rare.

Physics 318
Problem Set 12
Due: Friday, April 28, 2006

1. Bernstein, Fishbane, and Gasiorowicz: Chapter 14 problem 10.
2. Bernstein, Fishbane, and Gasiorowicz: Chapter 14 problem 12.
3. Bernstein, Fishbane, and Gasiorowicz: Chapter 14 problem 14.