Physics 474
Problem Set 1
Due Friday Jan 16th, 2009

1. Categorize the following as bosons or fermions. As always, try to give reasons for your answers.
   (a) π^+ ("pi-plus",$u\bar{d}$)
   (b) Δ^{++} (uuu)
   (c) Deuterium atom
   (d) Tritium atom
   (e) ^{23}Mg nucleus
   (f) ^{23}Mg^{2+} ion

{12}

2. Draw an energy level diagram, with a vertical axis like that of Fig 1.1 in the textbook but with a logarithmic energy scale extending from 0.1 MeV to 10^6 MeV. Plot six labelled horizontal lines on it, showing the masses of the six types of quark. Add to your diagram labelled lines for the three charged leptons. Add a final line showing the lower limit on the mass of the Higgs boson. **{8}**

Physics 474
Problem Set 2
Due Friday Jan 23, 2009

1. Show that the Dirac equation is the "square root" of the Klein-Gordon equation. Specifically, write the Dirac equation as

$$\mathsf{D}\psi = \frac{\partial}{\partial t}\psi$$

where D, the "Dirac operator", is a 4×4 matrix of derivative operators and constants. Show that acting with the Dirac operator twice ("squaring the Dirac equation") gives

$$\mathsf{D}^2\psi = \frac{\partial^2}{\partial t^2}\psi$$

and by using the explicit form of D given in class (and below) show that this is just the Klein-Gordon equation.

NB: Dirac equation is eqns (1.7) and (1.8) (using the *first* equality in (1.8)), with

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is the 2×2 zero matrix and σ_i are the Pauli spin matrices, eqn (5.21b).

Physics 474
Problem Set 3
Due Friday Jan 30th, 2009

1. Draw the lowest-order Feynman diagram for "light-light scattering", i.e. a process with two photons in the initial state, and two photons in the final state. Connecting them there should be interactions that allow them to exchange momentum. What power of α is the rate for this process proportional to? **{4}**

2. The strong nuclear interaction has an attractive part mediated by the exchange of pions ($M_\pi = 140$ MeV), and an (intrinsically stronger) repulsive part mediated by rho mesons ($M_\rho = 770$ MeV). What is the approximate range of the attractive part, and of the repulsive part (in fm)? Sketch the potential of the strong nuclear interaction as a function of distance. **{5}**

3. Express the following in "natural" ($\hbar = c = \varepsilon_0 = k_B = 1$) units, using MeV as your energy unit. **{8}**

   (a) The volume of a proton (taking its radius to be 1 fm).

   (b) The time for light to cross a Hydrogen atom (ie travel twice the Bohr radius, $a_{\text{Bohr}} = 0.53 \times 10^{-10}$ m).

   (c) Atmospheric pressure (10^5 Nm^{-2}).

   (d) A magnetic field of 1 Tesla.

4. Express the following in SI units. **{8}**

   (a) The mass of the pion, $m_\pi = 140$ MeV (convert to kg).

   (b) A flux of particles $J = 2.55 \times 10^{-25}$ MeV^3 (convert to m^{-2}s^{-1}).

   (c) The approximate energy density of a nucleus, 5×10^8 MeV^4 (convert to Jm^{-3}).

   (d) A temperature of 0.001 MeV (convert to Kelvin).

Physics 474
Problem Set 4
Due Friday Feb 6th, 2009

1. For each of the following processes, write down all the leading-order (in powers of α and α_W) Feynman diagrams. If the process cannot occur in the standard model (is "forbidden"), explain why by saying which conservation laws it breaks.

   (a) $e^+ \, e^- \to \mu^+ \, \mu^-$

   (b) $e^- \, \nu_\mu \to \mu^- \, \nu_e$

   (c) $\gamma \, \gamma \to \nu_e \, \nu_\mu$

   (d) $W^+ \to e^+ \, \bar{\nu}_e$

   (e) $\tau^+ \, \nu_\tau \to W^+ \, \gamma$

   {10}

2. The leptons are all pointlike particles, with no substructure that we can detect. Calculate the magnetic moment of each of the six leptons, using natural units with MeV as the unit of energy. **{7}**

3. Show how the universality of weak interactions across the 3 families of leptons leads to a relationship between the branching ratios $BR(\tau^- \to e^- + \text{neutrinos})$ and $BR(\mu^- \to e^- + \text{neutrinos})$. I.e., find an equation that relates these two branching ratios to each other. The equation may involve other properties of the τ and μ particles such as their lifetimes. Verify that the measured values are consistent with this prediction. **{8}**

Physics 474
Problem Set 5
Due Friday Feb 13th, 2009

1. In class we derived the amplitude $B(t)$ of the muon neutrino component in a propagating neutrino of energy E that started as an electron neutrino at time $t = 0$. We found that if the ν_e-ν_μ mixing angle is θ (and there is no mixing with other flavors) then $B(t) = \sin\theta\cos\theta\,(\exp(-iE_2t) - \exp(-iE_1t))$. Show that this implies that the probability at time t for the neutrino to be observed as a muon neutrino is $\sin^2(2\theta)\sin^2(\frac{1}{2}(E_2 - E_1)t)$ (eqn (2.28) in the textbook). What value of the mixing angle θ gives the largest probability that the neutrino will be found to have changed flavor? (This is called "maximal mixing".) **{6}**

2. A nuclear reactor emits ν_e particles of energy 1.5 MeV. The neutrino has mixing angle θ with another neutrino flavor which we will call ν_s (it could be ν_μ or ν_τ or some other undiscovered particle). A detector located 20km away measures the flux of ν_e and finds it is 80% to 90% of the expected value if there were no mixing. Assuming maximal mixing, what is the estimated value, with error bar, of $\Delta m^2 \equiv m(\nu_s)^2 - m(\nu_e)^2$, in eV^2? (You can assume that $\Delta m^2 \lesssim (2\ \mathrm{eV})^2$.) If the mixing angle were smaller, would your estimate of Δm^2 go down or up? **{9}**

3. Draw a possible quark-flow diagram (i.e. Feynman diagram where gluons are not shown) for each of the following processes. In cases where an internal photon or $W^\pm$ or Z^0 is needed, show a diagram that represents the *dominant* contribution to the rate (e.g., don't show a weak interaction if an electromagnetic interaction will also work; don't show a process with two photons if a one-photon process will also work). **{10}**

   (a) $p\,p \to p\,p\,\pi^+\,\pi^-$

   (b) $\pi^-\,p \to \pi^+\,\pi^-\,n$

   (c) $\pi^-\,n \to p\,\pi^-\,\pi^-$

   (d) $\pi^- \to e^-\,\bar{\nu}_e$

   (e) $p\,p \to p\,p\,e^-\,e^+$

Physics 474
Problem Set 6
Due Friday Feb 20th, 2009

1. Classify the following processes into strong, electromagnetic, weak, or forbidden and justify your choices.
   (a) $p \to n\ e^+\ \nu_e$
   (b) $\pi^-\ p \to \pi^+\ \pi^-\ n$
   (c) $\gamma\ p \to \pi^+\ n$
   (d) $\nu_e\ p \to n\ e^+\ \nu_\mu$
   (e) $\nu_\mu\ n \to \mu^-\ p$
   (f) $\pi^0 \to e^+\ e^+\ e^-\ e^-$

   {12}

2. The π^+ decays via weak interactions

$$\pi^+ \to \mu^+\ \nu_\mu$$
$$\pi^+ \to e^+\ \nu_e$$

   Taking the neutrinos to be massless, calculate the energy (in MeV) of the neutrino and the velocity (as a fraction of c) of the charged lepton in each decay. Work in the center of mass frame. You can look up particle masses in appendix E of the textbook; treat the neutrinos as massless. Use natural units. **{5}**

3. (a) Consider a classical field theory with two real fields ϕ_1 and ϕ_2. The potential V is

$$V(\phi_1, \phi_2) = \tfrac{1}{2}m^2(\phi_1^2 + \phi_2^2) + \lambda(\phi_1^2 + \phi_2^2)^2$$

   Where m^2 and λ are both positive. Assembling the two fields into a "vector" $\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, show that the potential can be written in terms of $|\boldsymbol{\phi}|$. Make a sketch of the 2D plot of the potential as a function of $|\boldsymbol{\phi}|$ and of the 3D plot of the potential as a function of ϕ_1 and ϕ_2. How many classical ground states does the system have, and at what values of the fields do they occur? **{5}**

   (b) Now consider the case where m is imaginary, so we define a real parameter $\tilde{m}$ by $m = i\tilde{m}$, so m^2 becomes $-\tilde{m}^2$. Show that the potential can now be written

$$V(|\boldsymbol{\phi}|) = V_0 + V_2(|\boldsymbol{\phi}|^2 - \phi_c^2)^2$$

   Obtain expressions for the real constants V_0, V_2, and ϕ_c in terms of $\tilde{m}$ and λ. Make a sketch of the 2D plot of $V(|\boldsymbol{\phi}|)$ and of the 3D plot of $V(\phi_1, \phi_2)$. How many classical ground states does the system have, and at what values of the fields do they occur? **{5}**

Physics 474
Problem Set 7
Due Friday Feb 27th, 2009

1. Two particles that we will call "X^0" and "Y^-" are produced in collisions such as $K^- \ p \to \pi^0 \ X^0$ and $K^- \ p \to K^+ \ Y^-$. Deduce the baryon number, strangeness and charm of X^0 and Y^-, and their quark content. Draw quark flow diagrams for the reactions specified above that produce them. **{6}**

2. (a) Draw a possible quark flow diagram for the decay $K^+ \to \mu^+ \ \nu_\mu \ \gamma$. Its branching ratio is 0.005 as compared with 0.63 for $K^+ \to \mu^+ \ \nu_\mu$: explain this suppression.
   (b) Draw two possible quark flow diagrams for the decay $K^+ \to \pi^0 \ \pi^+ \ \gamma$, one of which is "quark-line-disconnected" (no quark line goes from the initial state to the final state) and the other is "quark-line-connected" (both initial-state quark lines go through to the final state). You can treat the π^0 as $u\bar{u}$. Does this process have the characteristic timescale of a strong, electromagnetic, or weak process? Justify your answer. **{9}**

3. (a) How long does it take a particle of mass m and momentum p to travel a distance L? (Give the exact answer: do not assume that the particle is non-relativistic, nor that it is ultra-relativistic.)
   (b) Now assume that the particle is ultrarelativistic, i.e. $p \gg m$, and expand your answer to part (a) to lowest order in m/p.
   (c) Consider two ultrarelativistic particles of mass m_1 and m_2 with the same momentum p, that travel the same distance L. What is the difference $t_1 - t_2$ between their times of flight?
   (d) A mysterious meson of momentum 5 GeV travels from one block of scintillator to another, 4 m away. To what accuracy δt must we measure its time of flight in order to be able to tell whether the meson is a π^+ or a K^+? (The units of your final answer should be seconds).

   {10}

Physics 474
Problem Set 8
Due Friday Mar 20, 2009

1. A beam of positrons with energy E is directed at a stationary target of electrons. How high does E need to be in order to produce Z^0 bosons in these collisions? **{8}**

2. Consider a beam of neutrinos (with flux J) passing through a target of area A and length L.

   (a) What are the dimensions of J, in both SI and natural units?

   (b) Suppose the rate at which the neutrinos interact with the target is W interactions per second. For any individual neutrino, what is the probability p that it will experience an interaction while passing through the target? Express the result in terms of W, J, A, L. Check that your result is dimensionless.

   (c) Assume that the target contains n nucleons per unit volume, and that the cross section for a neutrino to interact with a nucleon is σ. Express p in terms of σ.

   {8}

3. The cross-section for an individual neutrino of energy E_ν to interact with a nucleon (proton or neutron) is $\sigma_\nu = KE_\nu$ where $K = 2.5 \times 10^{-20}$ MeV^{-3}.

   (a) What is the nucleon interaction cross-section in fm^2 for a beam with energy $E = 200$ GeV?

   (b) A neutron beam of energy $E = 200$ GeV passes through 1 m of iron. What proportion of the neutrinos interact with the iron nuclei? (The density of iron is 7860 kg/m^3).

   {8}

Physics 474
Problem Set 9
Due Friday March 27, 2009

1. Consider an operator $\hat{T}$ in a system with Hamiltonian $\hat{H}$, obeying $[\hat{T},\hat{H}]=0$. Show that T is conserved, in the sense that the expectation value of T in any state does not change over time.
   The operator $\hat{T}$ generates a "$U(1)$" group whose elements are $\hat{g}(\alpha)=\exp(i\alpha\hat{T})$. Show that this is a symmetry group of the Hamiltonian, in the sense that if we transform all states in the theory using the same group element $\hat{g}$ then this does not change any of the matrix elements $\langle\psi|\hat{H}|\chi\rangle$. I.e., show that, for all α and for all states $|\chi\rangle$ and $|\psi\rangle$,

   $$\langle\psi'|\hat{H}|\chi'\rangle = \langle\psi|\hat{H}|\chi\rangle$$

   where $|\psi'\rangle$ and $|\chi'\rangle$ are the transformed states, $|\psi'\rangle \equiv \hat{g}(\alpha)|\psi\rangle$, $|\chi'\rangle \equiv \hat{g}(\alpha)|\chi\rangle$.

   {7}

2. The radial co-ordinate is $r=\sqrt{x^2+y^2+z^2}$. Show that $dr/dx=x/r$. Show that for a function f that only depends on distance r from the origin, $\hat{L}_i f(r)=0$ for $i=x,y,z$. This means that the angular momentum operators give zero when acting on any function that is spherically symmetric. Explain why one would have expected this, given that the angular momentum operators are also the generators of rotations. **{5}**

3. The angular momentum operators form an algebra:

   $$[\hat{L}_x,\hat{L}_y]=i\hat{L}_z \qquad [\hat{L}_y,\hat{L}_z]=i\hat{L}_x \qquad [\hat{L}_x,\hat{L}_z]=-i\hat{L}_y\ .$$

   Using these equations, show that the "total orbital angular momentum squared" operator $\hat{L}^2=\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$ commutes with $\hat{L}_z$. Explain why this means that it commutes with $\hat{L}_x$ and $\hat{L}_y$ as well.
   Which of the following Hamiltonians conserve orbital angular momentum:

   $$H_1=\frac{1}{mr^2}\hat{L}^2+V(r), \qquad H_2=\frac{1}{mr^2}\hat{L}^2+B_zL_z, \qquad H_3=\frac{1}{mr^4}\hat{L}^4+V(r)$$

   (where B_z is a constant). **{8}**

Physics 474
Problem Set 10
Due Friday April 3, 2009

1. For each of the following states, give the quantum number s for the combined spins of the constituents, the quantum number l for the combined orbital angular momenta of the constituents, and the quantum number j for the total angular momentum. Write the spectroscopic notation (e.g. ${}^4P_{5/3}$) for the state.

   (a) the pions

   (b) the ρ mesons

   (c) the nucleons N (i.e. p and n)

   (d) the Δ baryons

   {8}

2. The η meson has spin 0, and decays into three pions via an electromagnetic process. Obtain its parity. Explain why two-pion decays like $\eta^0 \rightarrow \pi^+ \pi^-$ or $\eta^0 \rightarrow \pi^0 \pi^0$ are not observed, and state (with reasons) whether their branching ratio is zero, or too small to measure. **{12}**

Physics 474
Problem Set 11
Due Friday Apr 10, 2009

1. State (with justification) which of the following particles have well-defined C-parity: γ, ν_e, Z^0, W^-, K^0, n. **{6}**

2. The ρ^0 sometimes decays, in about 10^{-20} s, to π^0 and γ. The ρ has $J^P = 1^-$ (i.e. spin 1 and parity -1), the π^0 has $J^P = 0^-$, and the γ has $J^P = 1^-$.

   (a) From the information given above, what can you say about the relative orbital angular momentum quantum number l of the π^0 and γ that are produced?

   (b) The π^0 has $C = +1$ and the γ has $C = -1$: what is the C-parity of the ρ^0?

   (c) Consider the decay $\rho^0 \to \pi^0\ \pi^0$. Can it occur via the electromagnetic interaction? Can it occur via the weak interaction?

   {14}

Physics 474
Problem Set 12
Due Friday Apr 17, 2009

1. The η^0 is an isospin singlet, and the pions form an isospin triplet. Illuminate the meaning of this in the following way. By performing a general isospin rotation simultaneously on all the constituent quarks, show that the π^+ can be isospin-rotated into a state that contains a component of π^0 and into a state that contains a component of π^-, but not into states that contain a component of η^0, where $\eta^0 \propto \bar{u}u + \bar{d}d$. **{8}**

2. Why is the mass of the $\Sigma^0(1193)$, whose quark content is *uds*, very similar to that of the $\Sigma^+(1190)$, which consists of *uus*, but different from that of the $\Lambda^0(1116)$, whose quark content is also *uds*? On the other hand, the lifetime of the Λ^0 is very similar to that of the Σ^+, but very different from that of the Σ^0: explain this. **{6}**

3. What is the quark content and lifetime of the Ω^-? Explain the rough order of magnitude of the lifetime: why is it much longer than that of the Σ^0? Draw a quark flow diagram (i.e. a Feynman diagram without gluons) for the decay $\Omega^- \rightarrow \Lambda^0\ K^-$.
   (Remember that "neutral currents", i.e. a quark changing flavor by emitting/absorbing a Z^0, do not occur. Only a $W^\pm$ can change the flavor of a quark.) **{6}**

Physics 474
Problem Set 13
Due Friday Apr 24, 2009

1. Write down an observable low-energy scattering process that could *only* occur via exchange of a Z^0, not via exchange of a W^+ or W^- or a photon. For the initial state, you can assume the availability of a beam of any kind of lepton, and a target of ordinary matter (electrons, protons, neutrons). For the final state, you should assume that all charged particles can be identified, and their momentum and energy measured, but neutral particles cannot be seen.
   {6}

2. A beam of K^0 particles is created at $x = 0$ and travels in the positive x direction. The K^0 particles have kinetic energy 2.5 MeV.

   (a) At $x = 1$ cm the beam hits a thin foil target. Just before entering the target, what proportion of the surviving kaons are K_S and what proportion are K_L?

   (b) In the target, the energy of the particles is not affected, but there is some absorption. The amplitude of the $\bar{K}^0$ component is reduced by 40%, and the amplitude of the K^0 component by 20%. On exiting the foil, what proportion of the surviving kaons are K_S and what proportion are K_L?

   (c) If you measure the CP of a particle in the beam, immediately after it leaves the target, what is the probability that you will conclude it is a K_S, and what is the probability that you will conclude it is a K_L?

   {10}

3. The D^0 meson is $\bar{u}c$ and the $\bar{D}^0$ is $\bar{c}u$. Show that the weak interaction induces D^0-$\bar{D}^0$ mixing, by writing down the Feynman diagram for the conversion of a D^0 to a $\bar{D}^0$.
   {6}