

Quantum Field Theory
Problem Set 1
Due Tues Sep 4th, 2007

1. Covariant notation.

Note that we use the metric $(-1, +1, +1, +1)$, so when we write things in components the results may differ from other textbooks which use $(+1, -1, -1, -1)$, e.g. Jackson's chapter on special relativity.

- (a) In $c = 1$ units, the derivative ∂_μ can be written in components as

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

What is ∂^μ in components?

- (b) The electromagnetic field strength tensor $F^{\mu\nu}$ can be written in components as a matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Write $F_{\mu\nu}$ as a matrix.

- (c) The charge and current form a 4-vector, $J^\mu = (\rho, \mathbf{J})$. What is J_μ ? Write the conservation of charge, $\dot{\rho} + \nabla \cdot \mathbf{J} = 0$, in covariant notation.
- (d) Write Maxwell's source equations, $\nabla \cdot \mathbf{E} = \rho$ and $\nabla \times \mathbf{B} - \dot{\mathbf{E}} = \mathbf{J}$, as a single covariant equation.

2. Matrices for boosts and rotations of 4-vectors.

- (a) Consider a general 4-vector $x^\mu = (ct, x, y, z)$. What does this vector become when it is rotated by angle $\delta\theta_z$ about the z -axis? Write down the corresponding Lorentz transformation as a 4×4 matrix that acts on x^μ .
- (b) Consider a general 4-vector $x^\mu = (ct, x, y, z)$. What does this vector become when it is boosted in the z direction with rapidity η_z ? Write down the corresponding Lorentz transformation as a 4×4 matrix that acts on x^μ .

3. Lorentz transformation.

In class we wrote a general infinitesimal Lorentz transformation as

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\beta})_\nu^\mu = \delta_\nu^\mu + \frac{i}{\hbar} \theta_k (J_k)_\nu^\mu + \frac{i}{\hbar} \beta_k (K_k)_\nu^\mu.$$

for $|\boldsymbol{\theta}|, |\boldsymbol{\beta}| \ll 1$, where $\boldsymbol{\theta}$ is the 3-vector $(\theta_x, \theta_y, \theta_z)$ and similarly for $\boldsymbol{\beta}$.

- (a) Using the explicit form of the boost operators given in class, show that, for $\boldsymbol{\theta} = 0$ and $\boldsymbol{\beta} = (0, 0, \epsilon)$ with $\epsilon \ll 1$, $\Lambda(\boldsymbol{\theta}, \boldsymbol{\beta})_\nu^\mu x^\nu$ is indeed an infinitesimal Lorentz boost in the z direction with rapidity ϵ .
 (You can do this by comparing with the result of the previous question, but note that it may look like a negative boost because of different conventions about whether you boost the system or the reference frame ("active" vs "passive" transformation)).

(b) Show that

$$\lim_{n \rightarrow \infty} (1 + x/n)^n = \exp(x)$$

(c) Construct a finite z -boost of rapidity β from n small boosts each of rapidity $\epsilon = \beta/n$. By taking the limit $n \rightarrow \infty$, show that the result is

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\beta})_{\nu}^{\mu} = \begin{pmatrix} \cosh \beta & 0 & 0 & \sinh \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \beta & 0 & 0 & \cosh \beta \end{pmatrix} .$$

4. Harmonic oscillator review.

The harmonic oscillator hamiltonian is

$$\hat{H} = \hbar\omega(\hat{a}^{\dagger} \hat{a} + \frac{1}{2}) ,$$

where

$$[\hat{a}, \hat{a}^{\dagger}] = 1 .$$

Assuming the existence of a ground state $|0\rangle$ obeying $\hat{a}|0\rangle = 0$, find the eigenvalues and eigenstates of the Hamiltonian. How do \hat{a}^{\dagger} and \hat{a} act on the eigenstates?

What operator measures the number of energy quanta in the system?

5. Textbook errata.

Go to web site <http://www.physics.ucsb.edu/~mark/qft.html>, the list of errata for the textbook, and make the corrections in your copy up to page 100 (or further). Note somewhere (e.g. the inside cover) the date on which you did this and the page range covered. Since the corrections are also available arranged by date, you will later be able to quickly find new corrections for these pages.

Quantum Field Theory
Problem Set 2
Due Fri Sep 14th, 2007

1. Creation operator formalism for n non-relativistic identical particles

Derive the Schrödinger equation for n non-relativistic identical particles that interact with each other, Srednicki (1.30), from the Hamiltonian for this system in creation operator notation, Srednicki (1.32), using definitions (1.31),(1.33).

2. Commutation relations of fields

Derive the commutation relations for the creation and annihilation operators, Srednicki (3.29), from the equal-time commutation relations for the quantum field $\varphi(\mathbf{x})$ and its conjugate momentum $\Pi(\mathbf{x})$ (3.38).

3. Complex scalar field

Srednicki Q. 3.5

Quantum Field Theory
Problem Set 3
Due Friday Sep 21st, 2007

1. Path integral formulation of single-particle quantum mechanics

Srednicki Q. 6.1. In (6.1a), start with (6.7) as the definition of the path integral, and assume that the Hamiltonian H is $P^2/(2m) + V(Q)$. You can then do the p_j integrals and obtain the constant C that the question asks for. In (6.1c), assume that $V(Q) = 0$, as in (6.1b).

2. Fourier-transformed path integral formulation of harmonic oscillator

Derive Srednicki (7.5) from (7.3), using (7.4).

3. Two-point correlation function of harmonic oscillator

Srednicki Q. 7.3.

Quantum Field Theory
Problem Set 4
Due Friday Sep 28th, 2007

1. Path Integral for free complex scalar field

Srednicki Q. 8.7

2. Retarded and Advanced Green's functions

Srednicki Q. 8.5.

Explanation: The Feynman propagator $\Delta(x - y)$ is defined by Srednicki (8.11), which includes an $i\epsilon$ term in the denominator. This term is the “pole prescription”: it ensures that when the integral over k^0 is performed, there is one pole in the lower half of the complex k^0 plane (which is where we must close the k^0 contour if $x^0 > y^0$) and one in the upper half (which is where we must close the k^0 contour if $x^0 < y^0$). However, for Δ_{ret} we want the integral to be zero if $x^0 > y^0$. Where should the poles be, to ensure this? What alternative expression should occur instead of $i\epsilon$ to obtain the desired positioning of the poles? (it will have some dependence on k^0). For more information, see *Brown*, p138-9, or *Nair*, Ch. 4.1, and remember that to turn the Feynman propagator into the retarded propagator you can *either* do an infinitesimal shift in the contour, as Nair does, *or* modify the infinitesimal shift in the poles, which is what Srednicki is asking for here.

3. Real φ^4 theory

Srednicki Q. 9.2. Ignore the counterterms in part (a) as well as part (b).

4. Complex φ^4 theory

Srednicki Q. 9.3. Ignore the counterterms in part (a) as well as part (b).

Quantum Field Theory
Problem Set 5
Due Friday Oct 5th, 2007

1. Feynman rules for complex φ^4 theory

Srednicki Q. 10.2.

Explanation: In Srednicki Q. 3.5 you found that this theory had two types of particle; a is the particle, with charge 1, and b is its antiparticle, with charge -1. The result of Srednicki Q. 5.1 was that the operator φ adds charge -1, i.e. it creates a b particle or destroys an a particle, whereas φ^\dagger adds charge 1, i.e. it creates an a particle or destroys a b particle. The result of Srednicki Q. 9.3 was that the propagators have arrows on them, indicating the direction of flow of charge.

Your answer should specify how the Feynman rules for this theory differ from those for φ^3 , including:

- (a) rules for specifying what kind of external line (i.e. which direction the arrow points) to use for incoming a , incoming b , outgoing a , and outgoing b particles, with given momenta.
- (b) the value (contribution to $i\mathcal{T}$) of each external line
- (c) the value of each internal line
- (d) the value of the interaction vertex, and what kind of lines can attach to it.
- (e) what counterterm vertices exist, and what their values are.

Srednicki's "more elegant approach" corresponds to always taking the momenta to point in the same direction as the charge arrows on the propagators. How does this affect the rules?

2. $\varphi\varphi \rightarrow \varphi\varphi$ scattering.

Consider $2 \rightarrow 2$ scattering in φ^3 theory, where there is only one type of particle, with mass m . Work in the center of mass frame.

- (a) Show that the Mandelstam variables t and u are given by (11.40) and (11.41).
- (b) Show that t and u are always negative.
- (c) Show that in the nonrelativistic limit of low-momentum scattering, the scattering amplitude (11.7) is given by (11.42).
- (d) Calculate the differential cross section $d\sigma/s\Omega$ in the nonrelativistic limit, and explain why Srednicki says that it is "almost isotropic".

3. Compton scattering.

Srednicki Q. 11.2.

4. Textbook errata.

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Quantum Field Theory
Problem Set 6
Due Friday Oct 12th, 2007

1. Conversion from SI to natural units

Express the following in natural ($\hbar = c = \epsilon_0 = k_B = 1$) units, using MeV as your energy unit.

- (a) The time for light to cross a Hydrogen atom (ie travel twice the Bohr radius, $a_{\text{Bohr}} = 0.53 \times 10^{-10}$ m).
- (b) The specific heat capacity of Gold, $0.127 \text{ J g}^{-1} \text{ K}^{-1}$.
- (c) The thermal conductivity of Silver, $4.18 \text{ W cm}^{-1} \text{ K}^{-1}$.
- (d) A magnetic field of 1 Tesla.

2. Conversion from natural to SI units

Express the following in SI units.

- (a) The mass of the pion, $m_\pi = 140 \text{ MeV}$ (convert to kg).
- (b) A flux of particles $J = 2.55 \times 10^{-25} \text{ MeV}^3$ (convert to $\text{m}^{-2}\text{s}^{-1}$).
- (c) The approximate energy density of a nucleus, $5 \times 10^8 \text{ MeV}^4$ (convert to Jm^{-3}).
- (d) A temperature of 0.001 MeV (convert to Kelvin).

3. Lehmann-Källén spectral decomposition of propagator

Srednicki Q. 13.1.

You know from Chapter 3 that the equal-time commutator between the field and its conjugate momentum is $[\varphi(x), \Pi(y)] = i\delta^3(\mathbf{x} - \mathbf{y})$, so once you have calculated $[\varphi(x), \dot{\varphi}(y)]$ you just need to relate $\Pi(y)$ to $\dot{\varphi}(y)$, and that is where Z_φ comes in.

Quantum Field Theory
Problem Set 7
Due Friday Oct 19, 2007

Remember that you can perform calculations using mathematical software tools such as *Mathematica*, as long as you provide printout that shows how you obtained your results.

1. One-loop self-energy in φ^3 theory

Derive Srednicki (14.36) from Srednicki (14.30).

2. Renormalization of the self-energy

In this question we will obtain the values of κ_A and κ_B (Srednicki (14.37) and (14.38)) by imposing the on-shell-scheme renormalization conditions $\Pi(-m^2) = \Pi'(-m^2) = 0$ on (14.39).

(a) Show that Srednicki (14.39) can be rewritten as

$$\begin{aligned} \Pi(k^2) = \alpha \left[\frac{1}{2} \int_0^1 dx D(x, k) \ln \left(\frac{D(x, k)}{D_0} \right) \right. \\ \left. + \frac{1}{2} k^2 \int_0^1 dx x(1-x) \ln(1-x+x^2) + \frac{1}{2} m^2 \int_0^1 dx \ln(1-x+x^2) \right. \\ \left. + \frac{1}{6} \kappa_A k^2 + \kappa_B m^2 \right] \quad (1) \end{aligned}$$

where D_0 is defined in (14.42).

(b) Evaluate the x integrals in the second line of eq. (1).

(c) Write down an expression for $\Pi(-m^2)$.

(d) Calculate $\Pi'(-m^2)$.

(e) Solve for κ_A and κ_B by setting $\Pi(-m^2) = \Pi'(-m^2) = 0$.

(f) Show that these values yield Srednicki (14.43).

(g) Show that this is equivalent to Srednicki (14.44).

3. Self-energy in real φ^4 theory

Calculate the self-energy to order λ for the theory of problem 9.2. As in φ^3 theory, there is one diagram containing a momentum loop, and one diagram from the Z_φ and Z_m renormalization counterterms. By imposing the on-shell renormalization conditions, obtain the values of A and B to order λ .

Quantum Field Theory
Problem Set 8
Due Friday Oct 26, 2007

1. Renormalizeability of theories of Dirac fields

Srednicki Q. 18.1.

2. Skeleton expansion in terms of proper vertices

For each of the diagrams in Fig. 9.13, show which tree-level diagram it comes from in the diagrammatic expansion of the quantum effective action.

3. Legendre Transform

- (a) Consider the non-relativistic Lagrangian for a free particle, $L(v) = \frac{1}{2}mv^2$. Show that its Legendre transform (based on $H = pv - L$) is $H(p) = p^2/(2m)$. Verify that $\partial H/\partial p = v$.
- (b) Consider the relativistic Lagrangian for a free particle, $L(v) = -m\sqrt{1 - v^2}$. What is its Legendre transform $H(p)$? Show that the relationship $\partial H/\partial p = v$ is equivalent to the standard result from relativistic kinematics, $p = mv\gamma$.
- (c) Consider a zero-temperature gas of identical free nonrelativistic spin- $\frac{1}{2}$ fermions. The fermion mass is m . Write down $E(N)$, the energy density as a function of the number density. (It may be helpful to express E and N in terms of the Fermi momentum p_F .) Calculate the Legendre transform, $P(\mu)$ (based on $P = \mu N - E$). This is the pressure as a function of the chemical potential. Show that $\partial P/\partial \mu = N$.

Quantum Field Theory
Problem Set 9
Due Friday Nov 2, 2007

1. Symmetries of the quantum effective action

Srednicki Q. 21.2

2. Particle number for complex scalar field

Derive Srednicki (22.19) from (22.17), for a free complex scalar field theory. Is this result consistent with interpreting the b -particles as antiparticles of the a -particles?

3. Energy/momentum conservation for a real scalar field

For a real scalar field theory

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$$

obtain the Noether current j^μ corresponding to symmetry under spacetime translation by a^ν . Obtain the stress-energy tensor $T^{\mu\nu}$ defined by $j^\mu = a_\nu T^{\mu\nu}$.

Quantum Field Theory
Problem Set 10
Due Friday Nov 9, 2007

1. Space-time translation of fields

The space-time translation operator, which translates a quantum state by a^μ , is $T(a)$: $|\psi\rangle \rightarrow T(a)|\psi\rangle$ for all $|\psi\rangle$.

- (a) Show that the action of $T(a)$ on any field operator $A(x)$ is $T(a)^{-1}A(x)T(a) = A(x - a)$.
- (b) Using the explicit form $T(a) = \exp(-iP^\mu a_\mu)$, and assuming an infinitesimal a^μ , show that $[A(x), P^\mu] = -i\partial^\mu A(x)$.
- (c) For the special case where $A(x)$ is the scalar field $\varphi(x)$ in a theory of free real scalar fields, write down the explicit form of P^μ in terms of $\varphi(x)$ and its conjugate momentum field $\Pi(x)$, and use the canonical commutation relations to verify the result proved in part (b).

2. $SO(N)$ symmetry of the N real scalar field theory

- (a) An infinitesimal $SO(N)$ rotation can be written as $R_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij}$. Show that if R preserves the magnitude and reality of any N -component vector that it acts on, then it must be an orthogonal matrix ($R^T = R^{-1}$), and that the generators T^a must be imaginary antisymmetric matrices.
- (b) Find the Noether currents $j^{a\mu}$ for the $SO(N)$ rotation symmetry of the Lagrangian (24.2). There is one for each generator of the symmetry group.
- (c) Using the canonical commutation relations for the N -component scalar field,

$$[\varphi_i(t, \mathbf{x}), \Pi_j(t, \mathbf{y})] = i \delta_{ij} \delta^3(\mathbf{x} - \mathbf{y}) ,$$

show that $[\varphi_i, Q^a] = (T^a)_{ij} \varphi_j$ where Q^a is the Noether charge, i.e. the integral over all space of j^{a0} .

Quantum Field Theory
Problem Set 11
Due Friday Nov 16, 2007

1. Imaginary part of the self-energy of an unstable field

Work through the derivation of (25.15) from (25.10). In particular,

- (a) Show that imposing the renormalization conditions $\text{Re}\Pi(-m_\varphi^2) = \text{Re}\Pi'(-m_\varphi^2) = 0$ leads to (25.12). Why is it legitimate to assume $k^2 < -4m_\chi^2$?
- (b) Make a plot (by hand or using mathematical software) of $D(x, k)$ for $0 < x < 1$ assuming $k^2 < 0$, and show that it is negative for $x_- < x < x_+$ where $x_\pm = \frac{1}{2} \pm \frac{1}{2}(1 + 4m_\chi^2/k^2)^{1/2}$.
- (c) Perform the relevant integral to obtain (25.15).

2. Physical mass vs. mass parameter in \overline{MS} scheme

Work through the derivation of (27.11) from (27.4). In particular,

- (a) Explain where (27.9) comes from.
- (b) Substitute (27.4) in to (27.9) and show that the result is (27.11).

Quantum Field Theory
Problem Set 12
Due Friday Nov 30, 2007

1. Beta function in φ^3 theory

Here we show that (28.20) is correct. This question is particularly straightforward if you use mathematical software to perform the algebra. Expand $d\alpha/d\ln\mu$ in powers of ϵ , up to order ϵ^2 ,

$$d\alpha/d\ln\mu = \beta_0 + \beta_1\epsilon + \beta_2\epsilon^2 .$$

- (a) Substitute this in to (28.19). By requiring that (28.19) is true for any ϵ , show that $\beta_0 = \alpha^2 G'_1(\alpha)$, $\beta_1 = -\alpha$, and $\beta_2 = 0$.
- (b) Show that this is still true if you expand $d\alpha/d\ln\mu$ to order ϵ^3 .
- (c) Obtain a relationship between $G'_1(\alpha)$ and $G'_2(\alpha)$.

2. Anomalous dimension of the mass and field in φ^3 theory

- (a) Starting with (28.4), (28.7), and (28.8), show how to obtain the anomalous dimension of the mass,

$$\gamma_m \equiv \frac{1}{m} \frac{dm}{d\ln\mu}$$

in terms of the coefficients $a_1(\alpha)$ and $b_1(\alpha)$. What does this evaluate to in 6-dimensional φ^3 theory?

- (b) Starting with (28.3) and (28.30)-(28.32), show how to obtain the anomalous dimension of the field

$$\gamma_\varphi = \frac{1}{2} \frac{d\ln Z_\varphi}{d\ln\mu}$$

in terms of the coefficient $a_1(\alpha)$. What does this evaluate to in 6-dimensional φ^3 theory?

3. Beta-function and anomalous dimensions in real φ^4 theory

Srednicki Q. 28.1. You may want to look ahead to Ch. 31 to obtain Z_φ , Z_m , and Z_λ .

Quantum Field Theory
Problem Set 13
Due Friday Dec 7, 2007

1. Reparameterization invariance of first two beta-function coefficients

Srednicki Q. 29.1. In part (b), use the most general forms, $dg_n/d\ln\mu = b_{nij}g_i g_j + d_{nij}g_i g_j g_k$ and $\tilde{g}_i = g_i + c_{ijk}g_j g_k$.

2. Expanding around the vacuum in a theory with broken discrete symmetry

By writing $\varphi(x) = v + \rho(x)$ and setting Z -factors to 1 in (31.1), obtain the value of v which ensures that the minimum of the potential is at $\rho = 0$. Now reintroduce the Z -factors and set $\lambda \rightarrow \lambda\tilde{\mu}^\epsilon$ in V , and derive the correct version of (31.14) and verify the typo in that equation in the book. This is particularly straightforward using mathematical software tools such as *Mathematica*.

3. Loop corrections in a theory with broken discrete symmetry

(a) Derive (31.18) from (31.15),(31.16),(31.17), and (31.4).

(b) Show that $\langle 0|\rho(x)|0\rangle$ has no $1/\epsilon$ divergences.

(c) Show that $\langle 0|\rho(x)|0\rangle = 0$ implies (31.19).