Quantum Field Theory II
Problem Set 1
Due Fri Jan 18th, 2008

1. $\underline{SO(3,1) \sim SU(2)_L \times SU(2)_R}$.

   Show that the algebra of the generators of $SU(2)_L \times SU(2)_R$, (33.18)-(33.20), follows from the algebra of the generators of $SO(3,1)$, (33.11)-(33.13), when one relates them via (33.16) and (33.17).

2. Textbook errata.

   Go to web site `http://www.physics.ucsb.edu/~mark/qft.html`, the list of errata for the textbook, and make the corrections in your copy up to page 300 (or further). Note somewhere (e.g. the inside cover) the date on which you did this and the page range covered. Since the corrections are also available arranged by date, you will later be able to quickly find new corrections for these pages.

Quantum Field Theory II
Problem Set 2
Due Fri Jan 25, 2008

1. Two spinors make a 4-vector.

   In this question we will use the "exponential map" to construct non-infinitesimal group operations. This will just involve taking the exponential of a diagonal matrix. Consider the operation "rotate by 180° about the z axis".

   (a) The corresponding element of the Lorentz group acting on a 4-vector is

   $$\Lambda^{\alpha}{}_{\beta} = \exp(\tfrac{1}{2} i\omega_{\mu\nu} S_V^{\mu\nu})^{\alpha}{}_{\beta}$$

   where S_V are the generators of the Lorentz group in the 4-vector representation (see last semester's first lectures, or Srednicki Q. 2.9). Write down the 4×4 matrix Λ, just based on what you know this rotation should do to a 4-vector. What is the value of the matrix of parameters $\omega_{\mu\nu}$ for this operation?

   (b) To act with the same operation on a left handed $(2,1)$ spinor, or a right-handed $(1,2)$ spinor, we use the *same* Lorentz parameters ω, and write down

   $$\begin{aligned} L^{a}{}_{b} &= \exp(\tfrac{1}{2} i\omega_{\mu\nu} S_L^{\mu\nu})^{a}{}_{b} \\ R^{\dot{a}}{}_{\dot{b}} &= \exp(\tfrac{1}{2} i\omega_{\mu\nu} S_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} \end{aligned}$$

   Using the expressions given in Ch. 35 for S_L and S_R, obtain the matrices L and R for rotation by 180° about the z axis.

   (c) Show that if we act with these matrices on the LH spinor ψ^b and the RH spinor $\chi^{\dagger\dot{b}}$, then the effect on a vector constructed from them, $V^\mu = \sigma^\mu_{b\dot{b}}\psi^b\chi^{\dagger\dot{b}}$, is the same as acting directly on V^μ with the Λ matrix from part (a).

Quantum Field Theory II
Problem Set 3
Due Fri Feb 1st, 2008

1. Dirac spinors.

   (a) Using the properties (36.8) of the spinor matrices, show that the gamma matrices as defined in (36.7) obey the Clifford algebra anticommutation relations,

   $$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

   (b) Show that for Dirac spinors with lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, the equation of motion is $(i\gamma^\mu \partial_\mu - m)\psi = 0$, which is equivalent to (36.18).

   (c) Show that the gamma matrices obey

   $$\mathcal{C}^{-1}\gamma^\mu \mathcal{C} = -(\gamma^\mu)^T$$

   where the charge conjugation matrix $\mathcal{C}$ is defined in (36.37).

2. Numerical gamma-matrices for testing expressions.

   Using a mathematical manipulation program, explicitly construct the γ matrices as numerical 4×4 matrices. E.g. in mathematica one creates a list "`gamma`" containing $\{\gamma^1, \gamma^2, \gamma^3, \gamma^0\}$,

```
gamma=Table[Null,{4}]; (* Setting up the list *)
gamma[[1]]={{0,0,0,1},{0,0,1,0},{0,-1,0,0},{-1,0,0,0}};
 (* similarly for gamma[[2]], gamma[[3]], gamma[[4]] *)
gamma5 = I gamma[[4]].gamma[[1]].gamma[[2]].gamma[[3]]; (* eqn (38.25) *)
```

   etc; Check that your numerical gamma matrices obey obey (47.1)-(47.3).

3. Conjugate momentum and anticommutation relations.

   Show that for Dirac spinors the conjugate momentum field is $\pi(x) = i\bar{\psi}(x)\gamma^0$ and for Majorana spinors it is $\pi(x) = \frac{1}{2}i\psi^T(x)\mathcal{C}\gamma^0$.
   Starting from the anticommutation relations (37.4) and (37.5) for Weyl spinors, show that for Dirac spinors the anticommutation relations are

   $$\begin{aligned}\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} &= 0 \\ \{\psi_\alpha(\mathbf{x},t), \bar{\psi}_\beta(\mathbf{y},t)\} &= \gamma^0_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\end{aligned}$$

   and for Majorana spinors they are

   $$\begin{aligned}\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} &= (\mathcal{C}\gamma^0_{\alpha\beta})_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \\ \{\psi_\alpha(\mathbf{x},t), \bar{\psi}_\beta(\mathbf{y},t)\} &= \gamma^0_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\end{aligned}$$

Quantum Field Theory II
Problem Set 4
Due Fri Feb 8th, 2008

1. Boosts and rotations of Dirac and Majorana spinors.

   Show that under a Lorentz boost of rapidity η in direction $\hat{\mathbf{p}}$, a Dirac or Majorana spinor ψ is transformed to $\exp(i\eta\hat{\mathbf{p}}\cdot\mathbf{K})\psi$ where $K^j = \frac{1}{4}i[\gamma^j,\gamma^0]$.

2. Explicit expressions for $u(\mathbf{p})$ and $v(\mathbf{p})$.

   Srednicki Q. 38.1.

3. Properties of $u(\mathbf{p})$ and $v(\mathbf{p})$

   (a) For the case $m = 1.0$ GeV and $\mathbf{p} = (0, 0, 2.0)$ GeV, numerically evaluate $u(\mathbf{p})$ and $v(\mathbf{p})$.

   (b) Prove (38.23), and verify it by plugging in the numerical values from part (a) on the left and right sides.

   (c) Prove (38.28) from (38.23), and verify it by plugging in the numerical values from part (a) on the left and right sides.

Quantum Field Theory II
Problem Set 5
Due Fri Feb 15th, 2008

1. Numerical gamma-matrix expression for charge conjugation.

   Add the charge conjugation matrix $\mathcal{C}$ to your collection of numerical expressions for the gamma matrices. Numerically verify (38.34),(38.35),(38.36).

2. Charge conjugation of spinor fields.

   Starting with the transformation rules for fermion fields (40.42), show that for a Dirac field ψ, the bilinears $\bar{\psi}\psi$, $\bar{\psi}i\gamma_5\psi$, and $\bar{\psi}\gamma^\mu\gamma_5\psi$ are even under charge conjugation. Show that $\bar{\psi}\gamma^\mu\psi$ is odd, and explain why this is true, in terms of the physical meaning of $\bar{\psi}\gamma^\mu\psi$. What happens to these relations if we assume ψ is a Majorana field? What does that tell us about the value of $\bar{\psi}\gamma^\mu\psi$ in that case?

3. Work through the derivation of the fermion propagator.

   Prove the expression (42.12) for the free fermion propagator, using the expansion of the field operator in terms of creation/annihilation operators, i.e. starting with (42.1) to (42.7). Repeat the derivation for a Majorana fermion where there is only one creation/annihilation operator, so the field operator can create or destroy a particle, so you start with (42.17),(42.18) instead of (42.1), (42.2).

Quantum Field Theory II
Problem Set 6
Due Fri Feb 22nd, 2008

1. Grassmann numbers.

   Consider a complex-number-valued function f on a space of 3 Grassmann variables ψ_1, ψ_2, ψ_3, defined by

   $$f(\psi_1, \psi_2, \psi_3) = a + \psi_i\, \beta_i + \tfrac{1}{2}\psi_i\psi_j\, c\, \varepsilon_{ij} + \frac{1}{6}\psi_i\psi_j\psi_k\, \zeta\, \varepsilon_{ijk}\ .$$

   Which of a, β_i, c, ζ are Grassmanns and which are complex numbers? Calculate the derivative of f with respect to ψ_1 and show that it agrees with Srednicki (44.11).

2. Grassmann Gaussian integral.

   Verify Srednicki (44.16) for a 2×2 matrix J. Prove (44.26) for $n = 2$, i.e. a 2×2 antisymmetric matrix M. Use (44.16) to generalize this to a proof of (44.26) for arbitrary even n.

3. Fermionic functional integral from Grassmanns.

   Show how (44.1) can be obtained from (44.40). Write down explicitly what χ, $\chi^\dagger$, η, $\eta^\dagger$, M, and M^{-1} map on to.

Quantum Field Theory II
Problem Set 7
Due Fri Feb 29th, 2008

1. C,P, and T in scalar and pseudoscalar Yukawa theory.

   Srednicki Q. 45.1.

2. Real-space Feynman diagrams in Yukawa theory

   (a) Write down the real-space correlation function that is related to the pair-production scattering process, $e^- \; e^- \to e^- \; e^- \; e^+ \; e^-$. (Hint: see para after (45.7) for related examples.)

   (b) Write down the lowest-order real-space *connected* Feynman diagrams that contribute to this correlation function. (Hint: there are three different "skeletons", at order g^4, for each of which there are various re-orderings of the initial/final states.)

   (c) Calculate the relative signs of these diagrams, using the procedure described in the paragraph after eqn (45.10).

Quantum Field Theory II
Problem Set 8
Due Fri Mar 7th, 2008

1. Tree-level scattering amplitudes in Yukawa theory.

   Using the techniques described in Chapter 45,

   (a) Calculate the tree-level scattering amplitude $i\mathcal{T}$ for $e^+ \, e^+ \to e^+ \, e^+$.

   (b) Calculate the tree-level scattering amplitude $i\mathcal{T}$ for $\varphi \, \varphi \to e^+ \, e^-$.

2. Gamma-matrix manipulations.

   Prove (47.16), (47.20), and (47.21).

Quantum Field Theory II
Problem Set 9
Due Fri Mar 21st, 2008

1. Basis for 4×4 matrices.

   Srednicki 47.3.

2. Majorana fermions.

   Derive (49.8) from (49.7), or from the equivalent expression that was given in lectures.

3. Twistors.

   Srednicki Q. 50.1.

4. Textbook errata.

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Quantum Field Theory II
Problem Set 10
Due Fri Mar 28th, 2008

1. Self-energies in pseudoscalar Yukawa theory.

   (a) Show how the parameter κ_φ in the expression (51.25) for the scalar self-energy is fixed to the value (51.26) by imposing the renormalization condition $\Pi'(-M^2) = 0$.

   (b) Show that the fermion self-energy is given by (51.36), starting from (51.27).

2. Vertex correction in Yukawa theory.

   Calculate the Yukawa interaction vertex function $\mathbf{V}_Y(p', p)$ to one loop. First derive (51.47) from (51.38) and (51.39), then impose the renormalization condition $\mathbf{V}_Y(0,0) = ig\gamma_5$. Obtain an expression that depends on p^2, $p \cdot p'$, p'^2, m^2, M^2, and g. It should be independent of ε and the renormalization scale μ. You do not need to evaluate the integral over the three Feynman parameters for general p, p' but you will need to evaluate it at $p = p' = 0$.

Quantum Field Theory II
Problem Set 11
Due Fri Apr 4th, 2008

1. Anomalous dimensions in Yukawa theory.

   Srednicki Q. 52.1.

2. Gaussian Integrals

   (a) Calculate the Gaussian Integral for a single complex variable z,

   $$Z(m) = \int dz dz^* \exp(-m z^* z) \ .$$

   (b) Calculate the Gaussian Integral for n complex variables z_i, as a function of the $n \times n$ matrix M,

   $$Z(M) = \int \prod_{i=1}^{n} dz_i dz_i^* \ \exp(z_i^* M_{ij} z_j) \ .$$

Quantum Field Theory II
Problem Set 12
Due Fri Apr 11th, 2008

1. Electromagnetic field Lagrangian in Coulomb gauge

   Show that the Electromagnetic field Lagrangian (55.1) can be written in Coulomb gauge as

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\partial_j A_i \partial_j A_i - \tfrac{1}{2}\varphi\partial_i\partial_i\varphi - \rho\varphi + J_i A_i$$

2. Integrating out a quadratic non-dynamical degree of freedom

   Srednicki asserts (after (56.11)) that integrating out $A^0(x)$ just means taking $\bar{A}^0(x)$, the solution to the equation of motion for A^0, and substituting it in to the path integral. I.e., rewriting $\varphi \equiv A^0$,

$$\int D\varphi \exp(iS[\varphi]) = C\exp(iS[\bar{\varphi}]) \quad \text{where} \quad \frac{\delta S}{\delta\varphi}[\bar{\varphi}] = 0. \tag{1}$$

   C is a constant. Check that this is true when $S[\varphi]$ is quadratic. Write

$$S[\varphi] = \int \Big(\varphi(x)W(x-y)\varphi(y) + \varphi(x)Z(y)\delta^4(x-y)\Big)\, d^4x d^4y\ ,$$

   where W and Z are arbitrary functions, compute the functional integral, prove (1) and evaluate C.

Quantum Field Theory II
Problem Set 13
Due Fri Apr 18th, 2008

1. Compton scattering

   Srednicki 59.1.

Quantum Field Theory II
Problem Set 14
Due Fri Apr 25th, 2008

1. Electron self-energy in QED

   Derive (62.30),(62.31),(62.32) from (62.28), and then show how (62.33) follows.

2. Form factors and the vertex function of QED

   Srednicki 63.1. In part (a), "gauge invariance" means independence of the gauge parameter ξ that occurs in the full photon propagator (which could be attached to the vertex function).