1. $SO(3,1) \sim SU(2)_L \times SU(2)_R$.

Show that the algebra of the generators of $SU(2)_L \times SU(2)_R$, (33.18)-(33.20), follows from the algebra of the generators of $SO(3,1)$, (33.11)-(33.13), when one relates them via (33.16) and (33.17).

2. Textbook errata.

Go to web site http://www.physics.ucsb.edu/~mark/qft.html, the list of errata for the textbook, and make the corrections in your copy up to page 300 (or further). Note somewhere (e.g. the inside cover) the date on which you did this and the page range covered. Since the corrections are also available arranged by date, you will later be able to quickly find new corrections for these pages.
1. Two spinors make a 4-vector.

In this question we will use the “exponential map” to construct non-infinitesimal group operations. This will just involve taking the exponential of a diagonal matrix. Consider the operation “rotate by 180° about the z axis”.

(a) The corresponding element of the Lorentz group acting on a 4-vector is

$$\Lambda^\alpha_\beta = \exp\left(\frac{1}{2}i\omega_{\mu\nu}S_{\nu}^{\mu}\right)^\alpha_\beta$$

where $S_{\nu}$ are the generators of the Lorentz group in the 4-vector representation (see last semester’s first lectures, or Srednicki Q. 2.9). Write down the $4 \times 4$ matrix $\Lambda$, just based on what you know this rotation should do to a 4-vector. What is the value of the matrix of parameters $\omega_{\mu\nu}$ for this operation?

(b) To act with the same operation on a left handed $(2,1)$ spinor, or a right-handed $(1,2)$ spinor, we use the same Lorentz parameters $\omega_{\mu\nu}$, and write down

$$L^a_b = \exp\left(\frac{1}{2}i\omega_{\mu\nu}S_L^{\mu\nu}\right)^a_b$$

$$R^a_b = \exp\left(\frac{1}{2}i\omega_{\mu\nu}S_R^{\mu\nu}\right)^a_b$$

Using the expressions given in Ch. 35 for $S_L$ and $S_R$, obtain the matrices $L$ and $R$ for rotation by 180° about the z axis.

(c) Show that if we act with these matrices on the LH spinor $\psi^b$ and the RH spinor $\chi^{\dagger b}$, then the effect on a vector constructed from them, $V^\mu = \sigma^\mu_{bb}\psi^b\chi^{\dagger b}$, is the same as acting directly on $V^\mu$ with the $\Lambda$ matrix from part (a).

2. Srednicki Q. 34.4.
1. Dirac spinors.

(a) Using the properties (36.8) of the spinor matrices, show that the gamma matrices as defined in (36.7) obey the Clifford algebra anticommutation relations,

$$\{\gamma^\mu, \gamma^\nu\} = -2g^\mu\nu$$

(b) Show that for Dirac spinors with lagrangian \( \mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m)\psi \), the equation of motion is \( (i\gamma^\mu \partial_\mu - m)\psi = 0 \), which is equivalent to (36.18).

(c) Show that the gamma matrices obey

$$C^{-1}\gamma^\mu C = -\left(\gamma^\mu\right)^T$$

where the charge conjugation matrix \( C \) is defined in (36.37).

Useful fact: \( \bar{\sigma}^\mu_{\dot{a}a}\sigma^\mu_{\dot{a}a} = \sigma^\mu_{\dot{a}a} \sigma^\mu_{\dot{a}a} \) (35.19).

2. Conserved Noether charge for Dirac spinors.

Show that the Noether current associated with the \( U(1) \) symmetry of the Dirac spinor lagrangian is \( j^\mu = \bar{\psi} \gamma^\mu \psi \), and that this is equivalent to \( \chi^\dagger \bar{\sigma}^\mu \chi - \xi^\dagger \bar{\sigma}^\mu \xi \).

3. Conjugate momentum and anticommutation relations.

Show that for Dirac spinors the conjugate momentum field is \( \pi(x) = i\bar{\psi}(x)\gamma^0 \) and for Majorana spinors it is \( \pi(x) = \frac{i}{2}\bar{\psi}^T(x)C\gamma^0 \).

Starting from the anticommutation relations (37.4) and (37.5) for Weyl spinors, show that for Dirac spinors the anticommutation relations are

$$\{\psi_\alpha(x,t), \psi_\beta(y,t)\} = 0$$
$$\{\psi_\alpha(x,t), \bar{\psi}_\beta(y,t)\} = \gamma_0^{\alpha\beta}\delta^3(x-y)$$

and for Majorana spinors they are

$$\{\psi_\alpha(x,t), \psi_\beta(y,t)\} = (C\gamma_0^{\alpha\beta})_{\alpha\beta}\delta^3(x-y)$$
$$\{\psi_\alpha(x,t), \bar{\psi}_\beta(y,t)\} = \gamma_0^{\alpha\beta}\delta^3(x-y)$$

Optional exercise: Numerical gamma-matrices for testing expressions.

Using a mathematical manipulation program, explicitly construct the \( \gamma \) matrices as numerical 4 \times 4 matrices. E.g. in mathematica one creates a list “\texttt{gamma}” containing \( \{\gamma^1, \gamma^2, \gamma^3, \gamma^0\} \),

\[
\texttt{gamma} = \text{Table[Null, \{4\}]; (* Setting up the list *)} \\
\texttt{gamma[[1]]} = \{\{0,0,0,1\}, \{0,0,1,0\}, \{0,-1,0,0\}, \{-1,0,0,0\}\}; \\
\texttt{(* similarly for gamma[[2]], gamma[[3]], gamma[[4]] *))} \\
\texttt{gamma5 = I gamma[[4]].gamma[[1]].gamma[[2]].gamma[[3]]; (* eqn (38.25) *)} \\
\texttt{etc; Check that your numerical gamma matrices obey (47.1)-(47.3).}
Quantum Field Theory II  
Problem Set 4  
Due Fri Feb 10th, 2012

1. Explicit expressions for \( u(p) \) and \( v(p) \).
Srednicki Q. 38.1.

2. Properties of \( u(p) \) and \( v(p) \)
   (a) For the case \( m = 1.0 \text{ GeV} \) and \( p = (0, 0, 2.0) \text{ GeV} \), numerically evaluate \( u(p) \) and \( v(p) \).
   (b) Verify (38.23) and (38.28) by plugging in the numerical values from part (a) on the left and right sides. If you did the optional exercise in the previous homework then you can use that code to do this question.

3. Charge conjugation of spinor fields.
Starting with the transformation rules for fermion fields (40.42), show that for a Dirac field \( \psi \), the bilinears \( \bar{\psi} \psi, \bar{\psi} i\gamma_5 \psi \), and \( \bar{\psi} \gamma^\mu \gamma_5 \psi \) are even under charge conjugation. Show that \( \bar{\psi} \gamma^\mu \psi \) is odd, and explain why this is true, in terms of the physical meaning of \( \bar{\psi} \gamma^\mu \psi \). What happens to these relations if we assume \( \psi \) is a Majorana field? What does that tell us about the value of \( \bar{\psi} \gamma^\mu \psi \) in that case?

4. Work through the derivation of the fermion propagator.
Prove the expression (42.12) for the free fermion propagator, using the expansion of the field operator in terms of creation/annihilation operators, i.e. starting with (42.1) to (42.7). Repeat the derivation for a Majorana fermion where there is only one creation/annihilation operator, so the field operator can create or destroy a particle, so you start with (42.17), (42.18) instead of (42.1), (42.2).
Quantum Field Theory II  
Problem Set 5  
Due Fri Feb 17th, 2012

1. **Grassmann numbers.**

Consider a complex-number-valued function \( f \) on a space of 3 Grassmann variables \( \psi_1, \psi_2, \psi_3 \), defined by

\[
f(\psi_1, \psi_2, \psi_3) = a + \psi_i b_i + \frac{1}{2} \psi_i \psi_j c_{ij} + \frac{1}{6} \psi_i \psi_j \psi_k d_{ijk}.
\]

Which of \( a, b_i, c_{ij}, d_{ijk} \) are Grassmans and which are complex numbers? What can you say about the matrix \( c_{ij} \)? What can you say about \( d_{ijk} \)? Calculate the derivative of \( f \) with respect to \( \psi_1 \) and show that it agrees with Srednicki (44.11).

2. **Grassmann Gaussian integral.**

Verify Srednicki (44.16) for a \( 2 \times 2 \) matrix \( J \). Prove (44.26) for \( n = 2 \), i.e. a \( 2 \times 2 \) antisymmetric matrix \( M \). Use (44.16) to generalize this to a proof of (44.26) for arbitrary even \( n \).

3. **Fermionic functional integral from Grassmanns.**

Show how (44.1) can be obtained from (44.40). Write down explicitly what \( \chi, \chi^\dagger, \eta, \eta^\dagger, M, \) and \( M^{-1} \) map on to.

Note that in (44.1) there should be a constant multiplying the right hand side, chosen to ensure that \( Z[0]=1 \).

4. **C, P, and T in scalar and pseudoscalar Yukawa theory.**

Srednicki Q. 45.1.
Quantum Field Theory II  
Problem Set 6  
Due Fri Feb 24th, 2012

1. Real-space Feynman diagrams in Yukawa theory
   
   (a) Write down the real-space correlation function that is related to the pair-production scattering process, $e^- e^- \rightarrow e^- e^- e^+ e^-$. (Hint: see para after (45.7) for related examples.)

   (b) Write down the lowest-order real-space connected Feynman diagrams that contribute to this correlation function. (Hint: there are three different topologically different “skeletons”, at order $g^4$, for each of which there are various re-orderings of the initial/final states.)

   (c) Calculate the relative signs of these diagrams, using the procedure described in the paragraph after eqn (45.10).

2. Tree-level scattering amplitudes in Yukawa theory.

   Using the techniques described in Chapter 45,

   (a) Calculate the tree-level scattering amplitude $i\mathcal{T}$ for $e^+ e^+ \rightarrow e^+ e^+$.

   (b) Calculate the tree-level scattering amplitude $i\mathcal{T}$ for $\varphi \varphi \rightarrow e^+ e^-$. 
1. **Gamma-matrix manipulations.**
   Prove (47.16), (47.20), and (47.21).

2. **Basis for $4 \times 4$ matrices.**
   Srednicki 47.3.

3. **Self-energy in pseudoscalar Yukawa theory.**
   Show how the parameter $\kappa_\phi$ in the expression (51.25) for the scalar self-energy is fixed to the value (51.26) by imposing the renormalization condition $\Pi'(-M^2) = 0$. 
1. **Vertex correction in Yukawa theory.**
   Calculate the Yukawa interaction vertex function $V_Y(p', p)$ to one loop. First derive (51.47) from (51.38) and (51.39), then impose the renormalization condition $V_Y(0, 0) = ig\gamma_5$. Obtain an expression that depends on $p^2$, $p \cdot p'$, $p'^2$, $m^2$, $M^2$, and $g$. It should be independent of $\varepsilon$ and the renormalization scale $\mu$. You do not need to evaluate the integral over the three Feynman parameters for general $p, p'$ but you will need to evaluate it at $p = p' = 0$.

2. **Anomalous dimensions in Yukawa theory.**
   Srednicki Q. 52.1.
Quantum Field Theory II  
Problem Set 9  
Due Fri Mar 23rd, 2012

1. Gaussian Integrals

   (a) Calculate the Gaussian Integral for a single complex variable $z$,
   
   $$Z(m) = \int dzdz^* \exp(-mz^*z) .$$

   (b) Calculate the Gaussian Integral for $n$ complex variables $z_i$, as a function of the $n \times n$ matrix $M$ (assumed to be Hermitian with positive eigenvalues)
   
   $$Z(M) = \int \prod_{i=1}^{n} dz_i d z_i^* \exp(z_i^* M z_i) .$$

2. Effective potential of a scalar field

   Consider Srednicki’s example (53.1) of a complex scalar field $\chi$ coupled to a background real scalar field $\varphi$. Assuming that the background field $\varphi$ is independent of $x$, obtain the effective potential $\Gamma[\varphi]$ (given by (53.17) and (53.15)) as a function of $m^2 - g\varphi$ and an ultraviolet cutoff $\Lambda$. Assume that $m^2 > g\varphi$, and explain the physical significance of this assumption.

   Hint: transform to momentum space, sum the series to give a logarithm, Wick-rotate (14.16) the momentum integral to Euclidean space, introduce the cutoff and perform the integral. Discard terms that vanish as $\Lambda \to \infty$, and cutoff-dependent terms that do not depend on $m^2 - g\varphi$. For more help in evaluating the integral, see, for example, section 3.5 of chapter 5 of “Aspects of Symmetry” by Coleman. You may use software like Mathematica, as long as you provide a printout of your calculations.

3. Electromagnetic field Lagrangian in Coulomb gauge

   Show that the Electromagnetic field Lagrangian (55.1) can be written in Coulomb gauge as
   
   $$\mathcal{L} = \frac{1}{2} \dot{A}_i \dot{A}_i - \frac{1}{2} \partial_j A_i \partial_j A_i - \frac{1}{2} \varphi \partial_i \partial_i \varphi - \rho \varphi + J_i A_i$$
Quantum Field Theory II
Problem Set 10
Due Fri Mar 30th, 2012

1. Sourceless Maxwell equations

Show that the sourceless Maxwell equations (54.3) and (54.4) automatically follow from
the relationship between the field strength and the vector potential (54.10) (using (54.11)
and (54.12)).

2. Integrating out a quadratic non-dynamical degree of freedom

Srednicki asserts (after (56.11)) that integrating out $A^0(x)$ just means taking $\bar{A}^0(x)$, the
solution to the equation of motion for $A^0$, and substituting it in to the path integral.
I.e., rewriting $\varphi \equiv A^0$,

$$\int D\varphi \exp(iS[\varphi]) = C \exp(iS[\bar{\varphi}]) \text{ where } \frac{\delta S}{\delta \varphi}[\bar{\varphi}] = 0.$$ 

(1)

$C$ is a constant. Check that this is true when $S[\varphi]$ is quadratic. Write

$$S[\varphi] = \int \left( \varphi(x)W(x-y)\varphi(y) + \varphi(x)Z(y)\delta^4(x-y) \right) d^4x d^4y,$$

where $W$ and $Z$ are arbitrary functions, compute the functional integral, prove (1) and
evaluate $C$.

3. Derive Srednicki (57.3) from (57.2).
1. Compton scattering
   Srednicki 59.1.
1. **Electron self-energy in QED**
   Derive (62.30),(62.31),(62.32) from (62.28), and then show how (62.33) follows.

2. **Form factors and the vertex function of QED**
   Srednicki 63.1. In part (a), “gauge invariance” means independence of the gauge parameter $\xi$ that occurs in the full photon propagator (which could be attached to the vertex function).
1. Magnetic moment due to orbital angular momentum

Srednicki 64.1. To make the orbital angular momentum contribution clear, it is best to write the vector potential in a gauge that is invariant under rotations about the z-axis. Then the terms that arise from the $i\gamma^2 \partial_{\mu_1}$ term in (64.10) can be written in terms of $\hat{L}_z$. 


1. **Adjoint representation**

Show that the adjoint representation is indeed a representation of a Lie algebra. In other words, give all of the details of the steps between (70.3) and (70.8) in Srednicki.

2. **Three gluon vertex**

Derive the expression (72.5) for the three gluon vertex. Hint: You need to look at the tree level expression for the relevant correlation function; this you can derive using functional derivatives of the generating function of the free theory $Z_0(J)$.

3. **$\beta$ function and running of the coupling**

Starting from (73.35), go through the general analysis that leads to (73.39). Make sure to justify your steps by explaining how the different quantities should depend on $\mu$, $\epsilon$, etc. For example, why is $d\alpha_0/d\ln \mu = 0$? Hint: More details of the general analysis are given in chapter 28 of Srednicki, and you also did something similar in problem 2 of Problem Set 8.