Rules for manipulating Weyl spinor indices

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1. Epsilon symbols:

$$\varepsilon^{ab} = \varepsilon^{\dot a\dot b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \varepsilon_{ab} = \varepsilon_{\dot a\dot b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

   These are antisymmetric, so

$$\varepsilon^{ab} = -\varepsilon^{ba} \quad \varepsilon_{ab} = -\varepsilon_{ba} \quad \varepsilon^{\dot a\dot b} = -\varepsilon^{\dot b\dot a} \quad \varepsilon_{\dot a\dot b} = -\varepsilon_{\dot b\dot a} \tag{2}$$

2. Matrix product of two epsilon symbols is the identity:

$$\varepsilon^{ab}\varepsilon_{bc} = \varepsilon_{ab}\varepsilon^{bc} = \varepsilon^{\dot a\dot b}\varepsilon_{\dot b\dot c} = \varepsilon_{\dot a\dot b}\varepsilon^{\dot b\dot c} = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

   You can write this as

$$\begin{aligned} \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c = I \quad \varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c = I \\ \varepsilon^{\dot a\dot b}\varepsilon_{\dot b\dot c} = \delta^{\dot a}{}_{\dot c} = I \quad \varepsilon_{\dot a\dot b}\varepsilon^{\dot b\dot c} = \delta_{\dot a}{}^{\dot c} = I \end{aligned} \tag{4}$$

   as long as you remember a very important caveat: this "δ" symbol does *not* obey the usual index raising and lowering rules. See (9) below.

3. Matrix product of an epsilon and a transposed epsilon is *minus* the identity:

$$\begin{aligned} \varepsilon^{ab}\varepsilon_{cb} = \varepsilon^{ba}\varepsilon_{bc} = -\delta^a{}_c = -I \quad \varepsilon_{ab}\varepsilon^{cb} = \varepsilon_{ba}\varepsilon^{bc} = -\delta_a{}^c = -I \\ \varepsilon^{\dot a\dot b}\varepsilon_{\dot c\dot b} = \varepsilon^{\dot b\dot a}\varepsilon_{\dot b\dot c} = -\delta^{\dot a}{}_{\dot c} = -I \quad \varepsilon_{\dot a\dot b}\varepsilon^{\dot c\dot b} = \varepsilon_{\dot b\dot a}\varepsilon^{\dot b\dot c} = -\delta_{\dot a}{}^{\dot c} = -I \end{aligned} \tag{5}$$

4. To raise and lower indices use ε, where the *second* index of ε must contract with the index to be raised/lowered:

$$\begin{aligned} \chi^a = \varepsilon^{ab}\chi_b \quad \chi_a = \varepsilon_{ab}\chi^b \\ \xi^{\dagger\dot a} = \varepsilon^{\dot a\dot b}\xi^\dagger{}_{\dot b} \quad \xi^\dagger{}_{\dot a} = \varepsilon_{\dot a\dot b}\xi^{\dagger\dot b} \end{aligned} \tag{6}$$

   From (4) you can verify that raising an index and then lowering it again gives back the original quantity. This works similarly on symbols with more than one Weyl index, eg for a Weyl matrix L,

$$\begin{aligned} L_a{}^c = \varepsilon_{ab}\varepsilon^{cd}L^b{}_d \quad L^a{}_c = \varepsilon^{ab}\varepsilon_{cd}L_b{}^d \\ L_{\dot a}{}^{\dot c} = \varepsilon_{\dot a\dot b}\varepsilon^{\dot c\dot d}L^{\dot b}{}_{\dot d} \quad L^{\dot a}{}_{\dot c} = \varepsilon^{\dot a\dot b}\varepsilon_{\dot c\dot d}L_{\dot b}{}^{\dot d} \end{aligned} \tag{7}$$

5. If indices are explicitly specified then there is no ambiguity, but if indices are suppressed then you have to be careful about putting them back in. Implicit dotted indices are assumed to run *uphill*, implicit undotted indices are assumed to run *downhill*.

$$\chi\psi \equiv \chi^a\psi_a = -\chi_a\psi^a \quad \xi^\dagger\rho^\dagger \equiv \xi^\dagger{}_{\dot a}\rho^{\dagger\dot a} = -\xi^{\dagger\dot a}\rho^\dagger{}_{\dot a} \tag{8}$$

The last equalities follow from (6) and (5).

Non-standard transformation properties of δ defined in (4)

The Kroenecker delta symbol is not a Weyl tensor and does not obey raising and lowering rules. To see this just substitute $L^a{}_c = \delta^a{}_c$ in to (7) and use (5):

$$\delta^a{}_c \overset{?}{=} \varepsilon^{ab}\varepsilon_{cd}\delta_a{}^c = \varepsilon^{ab}\varepsilon_{ad} = -I \tag{9}$$

This seems to imply that $\delta^a{}_c = -I = -\delta_a{}^c$, contradicting (4)! You just have to remember that δ gets a minus sign whenever both its indices are raised/lowered, so

$$L_a{}^c = \delta_a{}^c + S_a{}^c \;\Rightarrow L^a{}_c = -\delta^a{}_c + S^a{}_c \tag{10}$$

This is really a notation issue, arising from our definition (4) of δ. Instead we could define an identity operator U that transforms in a standard way:

$$\begin{array}{ll} \mathrm{U}_a{}^c = I & \mathrm{U}^a{}_c = -I \\ \mathrm{U}^{\dot a}{}_{\dot c} = I & \mathrm{U}_{\dot a}{}^{\dot c} = -I \end{array} \tag{11}$$

So U is the identity Weyl operator, in the sense that

$$\mathrm{U}\chi = \chi \quad \mathrm{U}\xi^\dagger = \xi^\dagger \tag{12}$$

The only complication is that instead of (4) we would have

$$\begin{array}{ll} \varepsilon_{ab}\varepsilon^{bc} = \mathrm{U}_a{}^c & \varepsilon^{ab}\varepsilon_{bc} = -\mathrm{U}^a{}_c \\ \varepsilon^{\dot a\dot b}\varepsilon_{\dot b\dot c} = \mathrm{U}^{\dot a}{}_{\dot c} & \varepsilon_{\dot a\dot b}\varepsilon^{\dot b\dot c} = -\mathrm{U}_{\dot a}{}^{\dot c} \end{array} \tag{13}$$

So if you contract epsilons with undotted indices going downhill or dotted indices going uphill, you get $-\mathrm{U}$. But if the indices are contracted the other way you get $+\mathrm{U}$. The advantage of using U is that it acts exactly like any other Weyl tensor, so

$$L_a{}^c = \mathrm{U}_a{}^c + S_a{}^c \;\Rightarrow L^a{}_c = \mathrm{U}^a{}_c + S^a{}_c \tag{14}$$

So, for a Weyl matrix L acting on undotted (LH) indices, and R acting on dotted (RH) indices,

$$\begin{array}{c} \chi = L\psi \;\Rightarrow \chi_a = L_a{}^c\psi_c \;\Rightarrow \chi^a = -L^a{}_c\psi^c \\ \xi^\dagger = R\rho^\dagger \;\Rightarrow \xi^{\dagger\dot a} = R^{\dot a}{}_{\dot c}\rho^{\dagger\dot c} \;\Rightarrow \xi^\dagger{}_{\dot a} = -R_{\dot a}{}^{\dot c}\rho^\dagger{}_{\dot c} \end{array} \tag{15}$$

Note that Srednicki (34.12) defines the Lorentz transformations of spinors to be $\chi \to L\chi$ for left-handed spinors (34.1) and $\xi^\dagger \to -R\xi^\dagger$ for right-handed spinors (34.12).