Scalar Meson Spectroscopy with Lattice Staggered Fermions

Claude Bernard

*Physics Department, Washington University, St. Louis, MO 63130, USA*

Carleton DeTar and Ziwen Fu

*Physics Department, University of Utah, Salt Lake City, UT 84112, USA*

Sasa Prelovsek

*Department of Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia*

*and J. Stefan Institute, Jamova 39, Ljubljana, Slovenia*

(Dated: July 16, 2007)

Abstract

With sufficiently light up and down quarks the isovector (a₀) and isosinglet (f₀) scalar meson propagators are dominated at large distance by two-meson states. In the staggered fermion formulation of lattice quantum chromodynamics, taste-symmetry breaking causes a proliferation of two-meson states that further complicates the analysis of these channels. Many of them are unphysical artifacts of the lattice approximation. They are expected to disappear in the continuum limit. The staggered-fermion fourth-root procedure has its purported counterpart in rooted staggered chiral perturbation theory (rSχPT). Fortunately, the rooted theory provides a strict framework that permits the analysis of scalar meson correlators in terms of only a small number of low energy couplings. Thus the analysis of the point-to-point scalar meson correlators in this context gives a useful consistency check of the fourth-root procedure and its proposed chiral realization. Through numerical simulation we have measured correlators for both the a₀ and f₀ channels in the “Asqtad” improved staggered fermion formulation in a lattice ensemble with lattice spacing a = 0.12 fm. We analyze those correlators in the context of rSχPT and obtain values of the low energy chiral couplings that are reasonably consistent with previous determinations.

PACS numbers: 11.15.Ha, 12.38.Gc, 12.39.Fe, 14.40.Cs
I. INTRODUCTION

The recent evident successes of numerical simulations of QCD with improved staggered fermions demand a thorough examination of its most controversial ingredient, namely, using fractional powers of the determinant to simulate the correct number of quark species (the “fourth root trick”). The procedure is known to introduce nonlocalities and violations of unitarity at nonzero lattice spacing [1]. If these problems do not vanish in the continuum limit, they may even place the theory in an unphysical universality class. There are, however, strong theoretical arguments [2, 3, 4, 5] that the fourth-root trick is valid, i.e. that it produces QCD in the continuum limit.

One may also test the fourth-root procedure numerically. One can, for example, check that taste symmetry gets restored as the lattice spacing gets smaller, by looking at the eigenvalue spectrum [6, 7, 8, 9, 10], the Dirac operator [11], or the pion spectrum [12]. Alternatively, low-energy results of staggered fermion QCD simulations can be compared with predictions of rooted staggered chiral perturbation theory ($rS\chi$PT) [13, 14]. Since staggered chiral perturbation theory becomes standard chiral perturbation theory in the continuum limit, agreement between rooted QCD and ($rS\chi$PT) at nonzero lattice spacing would suggest that, at least for low energy or long-range phenomena, lattice artifacts produced by the fourth root approximation are as harmless as those produced by partial quenching. Partial quenching also induces unitarity violations, but they disappear in the limit of equal valence and sea quark masses.

There are two recent tests of agreement between rooted staggered fermion QCD and $rS\chi$PT: (1) Measurements of the light pseudoscalar meson masses and decay constants in partially quenched and full staggered fermion QCD fit well to expressions derived from $rS\chi$PT [15]. A byproduct of this fit is a determination of the low energy couplings of the theory. (2) The topological susceptibility measured in full QCD agrees reasonably well with predictions of $rS\chi$PT [16].

In the present work we examine scalar meson correlators in full QCD and compare their two-meson content with predictions of $rS\chi$PT. Since the appearance of the two-meson intermediate state is a consequence of the fermion determinant, an analysis of this correlator provides a direct test of the fourth root recipe. The $a_0$ channel has been studied recently in staggered fermion QCD by the MILC collaboration [17] and UKQCD collaboration [18].
Both groups found that the correlator appeared to contain states with energies well below possible combinations of physical mesons.

A simple explanation of the nonstandard features of the scalar correlators is provided by rSχPT [19, 20]. In that theory all pseudoscalar mesons come in multiplets of 16 tastes. The pattern of mass splittings is predicted by the theory. The $\pi$ and $K$ multiplets are split in similar ways. The $\eta$ and $\eta'$ mesons, however, are peculiar, because their masses are shifted by the axial $U(1)$ anomaly. Since the anomaly is a taste singlet, only the taste singlet $\eta$ and $\eta'$ acquire approximately physical masses. Some of the remaining members of the $\eta$ multiplet remain degenerate with the pions. According to taste symmetry selection rules, any two mesons coupling to a taste-singlet $a_0$ must have the same taste. But all tastes are equally allowed. Among other states, the taste singlet $a_0$ couples to the Goldstone pion (pseudoscalar taste) and an $\eta$, also with pseudoscalar taste and of the same mass. This spurious two-body state at twice the mass of the Goldstone boson accounts for the anomalous low-energy component in that channel.

This explanation raises concerns. Clearly, only the taste singlet $\eta$ approximates the physical state, since it is the only member of the multiplet subject to the anomaly. So if the other $\eta'$'s are not allowed as external states, we have violated unitarity in the sense that some intermediate states are not allowed as external states. Further examination of the taste multiplets in the intermediate states reveals that in addition to the several unphysical $\pi\eta$ taste combinations, there is a negative norm “ghost” contribution in the taste singlet $\eta$ meson leg [21]. Remarkably, all lattice artifacts resolve themselves in the continuum limit, however. The taste multiplets become degenerate, the two-body states merge, and the ghost state cancels the spurious taste combinations, leaving only the taste-singlet mesons. To achieve this cancellation requires following the rules of flavor counting in rSχPT.

In the present work we extend the analysis of Ref. [19, 20] and carry out a quantitative comparison of measured correlators and predictions of rSχPT. Despite the considerable complexity of channels with dozens of spectral components, the chiral theory models the correlators precisely in terms of only a small number of low energy couplings, which we may determine through fits to the data.

This article is organized as follows. Following a review of some needed results from SχPT in Sec II, we derive the chiral predictions for the $a_0$ and $f_0$ in Sec III. We present results of our fits to the predicted forms in Sec IV and conclude in Sec V.
II. ELEMENTS OF STAGGERED CHIRAL PERTURBATION THEORY

In this section we give a brief review of rooted staggered chiral perturbation theory with particular emphasis on the tree-level pseudoscalar mass spectrum. We obtain the rooted version of the theory through the replica trick, according to which each quark flavor, \( u, d, \) and \( s \), comes in four tastes and is repeated \( n_r \) times [22]. We calculate various quantities in the replicated theory, and in the final step, we set \( n_r = 1/4 \) to obtain the correct flavor counting.

The low energy effective chiral theory is formulated in terms of the meson field

\[
\Phi = \sum_{b=1}^{16} \frac{1}{2} T^b \phi^b
\]  

where \( T^b = \{1, \xi_5, i\xi_5\xi_\mu \ldots \} \) are Dirac gamma matrices and \( \phi^b \) is a \( 3n_r \times 3n_r \) matrix with rows and columns labeled by the flavor and replica index \( ur, dr, \) and \( sr \). The staggered chiral action is written in terms of the unitary matrix \( \Sigma = \exp(2i\Phi/f) \):

\[
S(\Sigma, m) = \int d^4y \left\{ \frac{f^2}{8} \text{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) - \frac{\mu f^2}{4} \text{Tr}(\mathcal{M} \Sigma^\dagger + M^\dagger \Sigma) + \frac{m^2_0}{2} \phi^2_{0I} + a^2\mathcal{V}(\Sigma) \right\}.
\]

The low energy couplings at this order are \( f, \mu \), and the quark mass matrix \( \mathcal{M} = I_t \otimes I_r \text{diag}(m_u, m_d, m_s) \), where \( I_t \) is the unit matrix in taste space and \( I_r \) is the unit matrix in replica space. The axial anomaly appears through the mass term \( m^2_0 \). It involves the flavor-singlet taste-singlet field \( \phi_{0I} = \sum_{f,r} \phi^I_{fr,fr}/\sqrt{3n_r} \). The taste-breaking term \( \mathcal{V} \) is a linear combination of operators [13, 14, 23]

\[
- \mathcal{V}(\Sigma) = \sum C_i \mathcal{O}_i,
\]

where

\[
\mathcal{O}_1 = \text{Tr} \left( T_{0.5} \Sigma T_{0.5} \Sigma^\dagger \right)
\]

\[
\mathcal{O}_{2V} = \frac{1}{4} [\text{Tr}(T_{0.5} \Sigma)\text{Tr}(T_{0.5} \Sigma^\dagger) + \text{h.c.}]
\]

\[
\mathcal{O}_{2A} = \frac{1}{4} [\text{Tr}(T_{0.5} \Sigma)\text{Tr}(T_{0.5} \Sigma^\dagger) + \text{h.c.}]
\]

\[
\mathcal{O}_3 = \frac{1}{2} [\text{Tr}(T_{0.5} \Sigma T_{0.5} \Sigma) + \text{h.c.}]
\]

\[
\mathcal{O}_4 = \frac{1}{2} [\text{Tr}(T_{0.5} \Sigma T_{0.5} \Sigma) + \text{h.c.}]
\]

\[
\mathcal{O}_{5V} = \frac{1}{2} [\text{Tr}(T_{0.5} \Sigma)\text{Tr}(T_{0.5} \Sigma^\dagger)]
\]
\[ \mathcal{O}_{5A} = \frac{1}{2} \left[ \text{Tr}(T_{0,\mu\Sigma}) \text{Tr}(T_{0,5\mu\Sigma}^\dagger) \right] \quad (10) \]
\[ \mathcal{O}_6 = \sum_{\mu<\nu} \text{Tr} \left( T_{0,\mu\nu} \Sigma T_{0,\nu\mu}^\dagger \right) . \quad (11) \]

Without the anomaly and taste-breaking term the tree-level masses of the pseudoscalar mesons with quark flavor content \( f, f' \) are, as usual,
\[ M^2_{f,f',b} = \mu (m_f + m_{f'}) . \quad (12) \]

The taste-breaking term splits the nonisosinglet states (\( \pi_b \) and \( K_b \)) to give
\[ M^2_{f,f',b} = \mu (m_f + m_{f'}) + a^2 \Delta_b , \quad (13) \]

where to leading order the multiplets split five ways,
\[ \Delta_5 = 0 \]
\[ \Delta_{\mu 5} = \frac{16}{f^2} (C_1 + 3C_3 + C_4 + 3C_b) \quad (14) \]
\[ \Delta_{\mu \nu} = \frac{16}{f^2} (2C_3 + 2C_4 + 4C_b) \]
\[ \Delta_\mu = \frac{16}{f^2} (C_1 + C_3 + 3C_4 + 3C_b) \]
\[ \Delta_I = \frac{16}{f^2} (4C_3 + 4C_4) , \]

which we label \( P, A, T, V, \) and \( I, \) respectively. This predicted multiplet pattern has been well confirmed in simulations [17, 24].

We will be working with degenerate \( u \) and \( d \) quarks (\( m_u = m_d = m_\ell \)), so it will be convenient to introduce the notation
\[ M^2_{Ub} = 2 \mu m_\ell + a^2 \Delta_b \]
\[ M^2_{Sb} = 2 \mu m_s + a^2 \Delta_b \]
\[ M^2_{Kb} = \mu (m_\ell + m_s) + a^2 \Delta_b \quad (15) \]

The isosinglet states (\( \eta \) and \( \eta' \)) are modified both by the taste-singlet anomaly and by the two-trace (quark-line hairpin) taste-vector and taste-axial-vector operators \( \mathcal{O}_{2V}, \mathcal{O}_{2A}, \mathcal{O}_{5V} \) and \( \mathcal{O}_{5A} \). When \( m_0^2 \) is large, in the taste-singlet sector we obtain the usual result
\[ M^2_{\eta,I} = \frac{1}{3} M^2_{U,I} + \frac{2}{3} M^2_{S,I} \]
\[ M^2_{\eta',I} = \mathcal{O}(m_0^2) . \quad (16) \]
In the taste-axial-vector sector we have

\[ M_{\eta A}^2 = \frac{1}{2}[M_{UA}^2 + M_{SA}^2 + 3n_r\delta_A - Z_A] \]
\[ M_{\eta' A}^2 = \frac{1}{2}[M_{UA}^2 + M_{SA}^2 + 3n_r\delta_A + Z_A] \]
\[ Z_A^2 = (M_{SA}^2 - M_{UA}^2)^2 - 2n_r\delta_A(M_{SA}^2 - M_{UA}^2) + 9n_r^2\delta_A^2, \]

where \( \delta_A = a^2\delta'_A = a^2 16(C_{2A} - C_{5A})/f^2 \), and likewise for \( A \to V \).

In the taste-pseudoscalar and taste-tensor sectors, in which is there is no mixing of the isosinglet states, the \( \eta_b \) and \( \eta'_b \) by definition have quark content \( \bar{u}u + \bar{d}d / \sqrt{2} \) and \( \bar{s}s \), respectively, and masses

\[ M_{\eta,b}^2 = M_{UB}^2 ; \quad M_{\eta',b}^2 = M_{SB}^2 \]  \hspace{1cm} (18)

In Table I we list the masses of the resulting taste multiplets for the lattice ensemble used in the present study with taste-breaking parameters \( \delta_A \) and \( \delta_V \) determined in Ref. [15, 17].

<table>
<thead>
<tr>
<th>( b )</th>
<th>( \pi_b )</th>
<th>( K_b )</th>
<th>( \eta_b )</th>
<th>( \eta'_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.1594</td>
<td>0.3652</td>
<td>0.1594</td>
<td>0.4927</td>
</tr>
<tr>
<td>A</td>
<td>0.2342</td>
<td>0.4036</td>
<td>0.1843</td>
<td>0.5129</td>
</tr>
<tr>
<td>T</td>
<td>0.2694</td>
<td>0.4250</td>
<td>0.2694</td>
<td>0.5384</td>
</tr>
<tr>
<td>V</td>
<td>0.2966</td>
<td>0.4428</td>
<td>0.2825</td>
<td>0.5491</td>
</tr>
<tr>
<td>I</td>
<td>0.3205</td>
<td>0.4592</td>
<td>0.4958</td>
<td></td>
</tr>
</tbody>
</table>

TABLE I: Masses of pseudoscalar meson taste multiplets in lattice units for the MILC coarse \( (a = 0.12 \text{ fm}) \) lattice ensemble \( \beta = 6.76, \ am_{ud} = 0.005, \ am_s = 0.05 \), as measured or inferred from measured masses and splittings. The mass of the \( \eta'_I \) depends on the anomaly parameter \( m_0 \).

III. SCALAR CORRELATORS FROM S\( \chi \)PT

In this section we rederive the “bubble” contribution to the \( a_0 \) channel of Ref. [20], using the language of the replica trick [13, 25], and then extend the result to the \( f_0 \) channel.

We match the point-to-point scalar correlators in chiral low energy effective theory and staggered fermion QCD by matching the Green’s functions, which are defined through the
generating functionals of the respective theories:

$$\frac{\partial^2 \log Z}{\partial m_{f,f'}(y) \partial m_{e,e'}(0)} . \tag{19}$$

For this purpose the quark mass term diag($m_u, m_d, m_s$) is converted to a local meson source $m_{f,f'}(y)$ (including flavor off-diagonal terms) in both S\(\chi\)PT and QCD.

A. Scalar correlator in staggered fermion QCD

First we review the construction of the needed correlators in staggered lattice QCD, where the generating functional is

$$Z(m_{f,f'}) = \int dU \exp[-S_g(U)] \det[M(U, m_{f,f'})]^{1/4} . \tag{20}$$

Here $U$ are the gauge link variables, $S_g(U)$ is the gauge action, and $M$ is the fermion matrix including flavor components. We work on a lattice of spacing $a$ and dimension $L^3 \times N_t$ and label sites by the integer four-vector $x_\mu$. Hypercubes of size $2^4$ are similarly labeled by $y_\mu$, so $x_\mu = 2y_\mu + \eta_\mu$.

Staggered fermion meson correlators can be defined in the one-component basis of the Grassman color vector field $\chi_f(x)$ or in the spin-taste basis of the field $q_f^{\alpha,a}(y)$ with spin label $\alpha$ and taste label $a$. The fields are related through

$$q_f^{\alpha,a}(y) = \frac{1}{8} \sum_\eta \Gamma_\eta^{\alpha,a} \chi_f(2y + \eta)$$

$$\chi_f(2y + \eta) = 2 \text{Tr}[\Gamma_\eta q_f(y)] \tag{21}$$

where $\Gamma_\eta = \gamma_0^{\eta_0} \gamma_1^{\eta_1} \gamma_2^{\eta_2} \gamma_3^{\eta_3}$, and the sum over $\eta$ runs over sites in the $2^4$ hypercube labeled by $y$. The lattice $y$ has spacing $A = 2a$.

For constructing the meson correlators via the functional derivative (19) we need to introduce the source term into Lagrangian

$$S_m = a^4 \sum_x \bar{\chi}_f(x) \chi_f(x) m_{f,f'}(x) . \tag{22}$$

To express the source in term of the spin-taste basis we use the relation

$$a^4 \bar{\chi}_f(2y + \eta) \chi_f(2y + \eta) = \frac{A^4}{16} \sum_\Gamma \zeta(\Gamma, \eta) \rho_{f,f',\Gamma}(y) \tag{23}$$
with
\[ \zeta(\Gamma, \eta) = \text{Tr}(\Gamma^\dagger \eta \Gamma^\dagger \eta \Gamma) / 4 \] (24)

and
\[ \rho_{f,f',\Gamma}(y) = \bar{q}_f(y) \Gamma \otimes \Gamma^* q_{f'}(y). \] (25)

The direct product \( \Gamma_S \otimes \Gamma_T^* \) acts on spin and taste components, respectively. So we obtain
\[ S_m = A^4 \sum_{y,\Gamma} \rho_{f,f',\Gamma}(y) m_{f,f',\Gamma}(y) \] (26)

with
\[ m_{f,f',\Gamma}(y) = \frac{1}{16} \sum_{\eta} \zeta(\Gamma, \eta) m_{f,f'}(2y + \eta) \] (27)

The desired source for the scalar density, \( m_{f,f',I}(y) \), has \( \Gamma = I \) and \( \zeta(I, \eta) = 1 \). It is the component of \( m_{f,f'}(x) \) that is constant over a \( 2^4 \) hypercube. The other terms \( m_{f,f',\Gamma}(y) \) are sources for the other local staggered mesons.

A particular correlator is obtained by differentiating the generating functional with respect to the appropriate source mass terms. The general two-point function is, then,
\[ \frac{\partial^2 \log Z}{\partial m_{f,f',\Gamma}(y) \partial m_{e,e',\Gamma'}(0)} \bigg|_{m_{f,f'}(x) = \delta_{f,f'} m_f} = A^8 \langle \rho_{f,f',\Gamma}(y) \rho_{e,e',\Gamma'}(0) \rangle. \] (28)

The above quantity will be calculated for \( \Gamma = I \) also within \( S\chi PT \) below.

Now, we need to relate the quantity (28) to the correlator generated from the code. In practice the simulated correlator is computed from a point source at the origin
\[ O_{e,e',\text{src}} = a^3 \bar{\chi}_{e'}(0) \chi_e(0) = \frac{A^3}{8} \sum_{\Gamma} \rho_{e,e',\Gamma}(0), \] (29)

and a single time-slice sink operator at time \( \tau = 2t + \eta_0 \),
\[ O_{f,f',\text{sink}}(\bar{y}, \tau) = a^3 \sum_{\bar{\eta}} \bar{\chi}_{f'}(2\bar{y} + \bar{\eta}, \tau) \chi_f(2\bar{y} + \bar{\eta}, \tau) = A^3 [\rho_{f,f',I}(2\bar{y}, t) + (-)^{\eta_0} \rho_{f,f',05}(2\bar{y}, t)], \] (30)

where we have used relation (23), \( \zeta(I, \eta) = 1 \) and \( \zeta(05, \eta) = (-)^{\eta_0} \). Note that the sink operator is defined on spatial cubes \( \bar{y} \) but all time slices \( \tau \).

In this language the computed correlator is
\[ C_{f,f';e,e'}(\vec{p}, \tau) = \sum_{\vec{y}} \exp(i \vec{p} \cdot \vec{y} A) \langle \bar{O}_{f,f';\text{sink}}(\bar{y}, \tau) O_{e,e',\text{src}} \rangle. \] (31)
The meson taste is conserved, so the correlator separates into nonoscillating and oscillating components for a taste-singlet scalar contribution and a taste-axial-vector pseudoscalar meson contribution, respectively.

\[ C_{f,f';e,e'}(\vec{p}, \tau a) = (\mathcal{C}_{f,f';e,e'}(\vec{p}, \tau a) + (\mathcal{C}_{f,f';e,e'}^{\text{osc}}(\vec{p}, \tau a), \quad (32) \]

where

\[ \mathcal{C}_{f,f';e,e'}(\vec{p}, \tau a) = \frac{A^6}{8} \sum_{\vec{y}} \exp(i\vec{p} \cdot \vec{y} A) \langle \rho_{f',e'}(2\vec{y}, t) \rho_{e,e'}(0) \rangle. \quad (33) \]

The correlator has a quark-line-connected part and may also have a quark-line disconnected part:

\[ C_{f,f';e,e'}(\vec{p}, \tau a) = C_{f,f';e,e'}^{\text{conn}}(\vec{p}, \tau a) + C_{f,f';e,e'}^{\text{disc}}(\vec{p}, \tau a) \quad (34) \]

The quark-line disconnected part appears only in the taste-singlet isosinglet correlator.

To compute the correlator we need to express it in terms of quark propagators. So we start from the definition of the correlator in Eq. (31), substitute the definitions of the operators in Eqs. (29) and (30) and use the relation

\[ a^8 \langle \bar{\chi}_f(2y + \eta) \chi_{f'}(2y + \eta) \bar{\chi}_{e'}(\eta') \chi_{e'}(\eta') \rangle = \left. \frac{\partial^2 \log Z}{\partial m_{f',f}(2y + \eta) \partial m_{e',e}(\eta') \bigg|_{m_{f',f}(x) = \delta f',m}} \right. \]

\[ = \frac{A^8}{256} \sum_{\Gamma,\Gamma'} \zeta(\Gamma, \eta) \zeta(\Gamma', \eta') \langle \bar{\rho}_{f',e'}(y) \rho_{e',e'}(0) \rangle, \quad (35) \]

which follows from identity

\[ \frac{\partial}{\partial m(2y + \eta)} = \frac{1}{16} \sum_{\Gamma} \zeta(\Gamma, \eta) \frac{\partial}{\partial m_\Gamma(y)}. \quad (36) \]

Finally we arrive at the point-to-point correlators

\[ C_{f,f';e,e'}^{\text{conn}}(\vec{p}, \tau a) = -\sum_{\vec{x}} (-)^x \exp(i\vec{p} \cdot \vec{x} a) \left\langle \text{Tr}[M^{-1}_f(\vec{x}, \tau; 0, 0) M^{-1}_0(\vec{x}, \tau; 0, 0)] \right\rangle \delta_{ef} \delta_{e'f'} \]

\[ C_{f,f';e,e'}^{\text{disc}}(\vec{p}, \tau a) = \frac{1}{4} \sum_{\vec{x}} \exp(i\vec{p} \cdot \vec{x} a) \left\langle \text{Tr}[M^{-1}_f(\vec{x}, \tau; \vec{x}, \tau)] \text{Tr}[M^{-1}_0(0, 0; 0, 0)] \right\rangle \delta_{ee'} \delta_{f'f}, \]

where we have used Eq. (20) and the normalization \( M = 2D + 2am \) for the Dirac matrix. We keep the momentum \( p \) small, so we can neglect variation of the exponential over the hypercube.

As it is computed, at zero momentum the quark-line disconnected correlator includes the vacuum disconnected piece:

\[ C_{f,e,0} = \frac{L^3}{4} \left\langle \text{Tr}[M^{-1}_f(0, 0; 0, 0)] \right\rangle \left\langle \text{Tr}[M^{-1}_0(0, 0; 0, 0)] \right\rangle. \quad (37) \]
B. Scalar correlator in $S_{\chi PT}$

The continuum generating functional for scalar correlators in $S_{\chi PT}$

$$Z_{\text{SXPT}}(m_{f f'}) = \int [d\Sigma] \exp[-S(\Sigma, m_{f f'})],$$

(38)

where $S(\Sigma, m_{f f'})$ is given by Eq. (2). We do not include explicit scalar meson fields in the chiral Lagrangian, but add their contributions in the final expressions. To match the functional derivatives (19) we approximate the continuum integration in the chiral theory with a sum over hypercubic volumes of size $A^4$ and differentiate with respect to a constant source inside that volume. In this case $m_{f f', J}(y) = m_{f f'}(y)$. The source is also constant over replicas of the same flavor. The space-time volume equals that of QCD, namely, $A^4(L/2)^3 N_t/2$ for $A = 2a$. We use the integer four vector $y$ to label the hypercubes in the chiral theory.

The functional derivative in $S_{\chi PT}$ is

$$\frac{\partial^2 \log Z_{\text{SXPT}}}{\partial m_{f f'}(y) \partial m_{e' e'}(0)} = A^8 \mu^2 \sum_{r,r'} \left\langle \text{Tr} t \left( \Phi^2(y) \right)_{fr,f'r} \text{Tr} t \left( \Phi^2(0) \right)_{er',e'r'} \right\rangle,$$

(39)

At tree level the action (2) has no explicit quark-antiquark scalar meson fields, but it generates the two-pseudoscalar-meson “bubble” terms in the correlator. The functional derivative (39) corresponds to (28) with $\Gamma = J$. We use $B$ to denote the bubble contribution corresponding to the correlator (33)

$$B_{f f', e e', J}(\vec{p}, t A) = A^6 \sum_{\vec{y}} \exp(i \vec{p} \cdot \vec{y} A) \left\langle \rho_{f f', J}(y) \rho_{e e', J}(0) \right\rangle$$

(40)

We introduce its time Fourier transform

$$B_{f f', e e', J}(p) = \sum_{t=0}^{N_t/2} \exp(ip0t A) B_{f f', e e', J}(\vec{p}, t A).$$

(41)

At tree level the vacuum expectation value reduces through Wick contractions to products of meson two-point functions. In momentum space we have, generically, the Euclidean correlator

$$\left\langle \phi(y) \phi(0) \right\rangle = \frac{1}{A^4(L/2)^3 (N_t/2)} \sum_k \exp(ik \cdot y A) \left\langle \phi(-k) \phi(k) \right\rangle$$

(42)

where $\langle \phi(-k) \phi(k) \rangle = 1/(k^2 + m^2)$. So

$$\left\langle \phi(y) \phi(0) \right\rangle = \frac{1}{A^3(L/2)^3} \sum_{\vec{k}} \frac{\exp[-E(\vec{k})t A + i\vec{k} \cdot \vec{y} A]}{2E(\vec{k})}$$

(43)
for $E(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$. In terms of momentum components, the general term in the correlator becomes

$$
B_{f,f';e,e'}(p) = \frac{A^6 \mu^2}{8} \sum_y \exp(ip \cdot yA) \sum_{g,s,r,b,g',s',r',b'} \left\langle \phi^b_{fr,gs}(\vec{y}, t) \phi^b_{gs,f'r'}(\vec{y}, t) \phi^{b'}_{er',g's'}(0) \phi^{b'}_{g's'e'r'}(0) \right\rangle
$$

$$
= \frac{\mu^2}{8(L/2)^3(Nr/2)A^2} \sum_k \sum_{g,s,r,b,g',s',r',b'} \left[ \left\langle \phi^b_{fr,gs}(-k) \phi^{b'}_{er',g's'}(k) \right\rangle \left\langle \phi^b_{gs,f'r'}(k - p) \phi^{b'}_{g's'e'r'}(p - k) \right\rangle + \left\langle \phi^b_{fr,gs}(-k) \phi^{b'}_{g's'e'r'}(k) \right\rangle \left\langle \phi^b_{gs,f'r'}(k - p) \phi^{b'}_{er',g's'}(p - k) \right\rangle \right].
$$

We have used the fact that the bubble term, by definition, does not include the vacuum disconnected piece corresponding to Eq. (37).

There are two types of two-point functions, namely, the connected two-point function for all tastes:

$$
\left\langle \phi^b_{gs,f'r}(-k) \phi^{b'}_{f'r',g's'}(k) \right\rangle_{\text{conn}} = \frac{\delta_{r,r'}\delta_{f,f'}\delta_{g,g'}\delta_{s,s'}}{k^2 + M^2_{fg,b}},
$$

and the additional disconnected contribution for the taste-singlet, taste-axial-vector, and taste-vector mesons. For the taste singlet it is

$$
\left\langle \phi^I_{gs,f'r}(-k) \phi^{I}_{f'r',g's'}(k) \right\rangle_{\text{disc}} = -\frac{\delta_{r,r'}\delta_{f,f'}\delta_{g,g'}\delta_{s,s'}}{3n_r} \frac{k^2 + M^2_{Sl}}{(k^2 + M^2_{UL})(k^2 + M^2_{UL})}.
$$

Here we have already decoupled the taste-singlet η' by taking $m_0^2 \to \infty$. The disconnected contribution for the taste-axial-vector meson is

$$
\left\langle \phi^A_{gs,f'r}(-k) \phi^{A}_{f'r',g's'}(k) \right\rangle_{\text{disc}} = -\delta_{r,r'}\delta_{f,f'}\delta_{g,g'}\delta_{s,s'} \frac{\delta_A(k^2 + M^2_{SA})}{(k^2 + M^2_{UL})(k^2 + M^2_{UL})(k^2 + M^2_{UL})},
$$

and there is a similar contribution for the taste-vector meson.

It is convenient to carry out a partial fraction expansion of the disconnected contributions as follows:

$$
\left\langle \phi^I_{gs,f'r}(-k) \phi^{I}_{f'r',g's'}(k) \right\rangle_{\text{disc}} = -\frac{\delta_{r,r'}\delta_{f,f'}\delta_{g,g'}\delta_{s,s'}}{3n_r} \left( \frac{3/2}{k^2 + M^2_{UL}} - \frac{1/2}{k^2 + M^2_{UL}} \right)
$$

and

$$
\left\langle \phi^A_{gs,f'r}(-k) \phi^{A}_{f'r',g's'}(k) \right\rangle_{\text{disc}} = -\delta_{r,r'}\delta_{f,f'}\delta_{g,g'}\delta_{s,s'} \left( \frac{g_U}{k^2 + M^2_{UL}} + \frac{g_8}{k^2 + M^2_{UL}} \frac{g_{8'}}{k^2 + M^2_{UL}} \right).
$$
where
\[
\begin{align*}
g_U &= \frac{M_{SA}^2 - M_{UA}^2}{(M_{\eta A}^2 - M_{UA}^2)(M_{\eta A}^2 - M_{\eta U}^2)} \\
g_\eta &= \frac{M_{SA}^2 - M_{\eta A}^2}{(M_{UA}^2 - M_{\eta A}^2)(M_{UA}^2 - M_{\eta U}^2)} \\
g_{\eta'} &= \frac{M_{SA}^2 - M_{\eta' A}^2}{(M_{UA}^2 - M_{\eta' A}^2)(M_{UA}^2 - M_{\eta' U}^2)}
\end{align*}
\]

In the language of Refs. [13, 14], \(g_U\), \(g_\eta\), and \(g_{\eta'}\) are simply the residues for Eq. (47). Similarly, the factors of \(3/2\) and \(-1/2\) in Eq. (48) are the residues for Eq. (46).

C. Isovector \(a_0\) correlator

We now specialize to the isovector \(a_0\) correlator. We consider, for simplicity, the \(u\bar{d}\) flavor state. Only the quark-line-connected contribution appears in the QCD correlator

\[
B_{a_0} (\vec{p}, \tau a) = B_{u,d;u,d} (\vec{p}, \tau a) = -\sum_x (-)^x \exp(i\vec{p} \cdot \vec{x}) \left\langle \Tr[M_{a}^{-1}(\vec{x}, \tau; 0, 0)M_{d}^{-1}(\vec{x}, \tau; 0, 0)] \right\rangle.
\]

In terms of the meson fields, the bubble correlator is (for \(\tau a = tA\))

\[
B_{u,d;u,d,t} (\vec{p}, tA) = \frac{A^6 \mu^2}{8} \sum_y \exp(i\vec{p} \cdot \vec{y}) \sum_{r,s,f,b} \sum_{r',s',f',b'} \left\langle \phi_{ur,fs}^b (\vec{y}, t) \phi_{fs,dr}^{b'} (\vec{y}, t) \phi_{dr',f's'}^{b''} (0) \phi_{f's',ur}^{b''} (0) \right\rangle.
\]

After carrying out the Wick contractions and switching to momentum space we get

\[
B_{a_0} (p) = \frac{\mu^2}{8A^2(L/2)^3(N_\tau/2)} \left\{ n_r^2 \sum_{f,b} \sum_k \frac{1}{k^2 + M_{fu,b}^2} \frac{1}{k^2 + M_{fu,b}^2} \right\} - 4n_r \sum_k \left[ \frac{1}{(k + p)^2 + M_{UL}^2} \frac{1}{3n_r (k^2 + M_{UL}^2)(k^2 + M_{\eta U}^2)} \right] - 4n_r \sum_k \left[ \frac{4\delta_A}{(k + p)^2 + M_{UL}^2} \frac{k^2 + M_{SA}^2}{k^2 + M_{UA}^2 (k^2 + M_{\eta A}^2)(k^2 + M_{\eta U}^2)(k^2 + M_{\eta' A}^2)(k^2 + M_{\eta' U}^2)} \right] - 4n_r \sum_k \left[ \frac{4\delta_V}{(k + p)^2 + M_{UL}^2} \frac{k^2 + M_{SV}^2}{(k^2 + M_{\eta' V}^2)(k^2 + M_{\eta V}^2)(k^2 + M_{\eta' V}^2)} \right].
\]
In the continuum limit, in which taste-symmetry is restored, we have

\[
B_{a_0}(p) = \frac{\mu^2}{8A^2(L/2)^3(N_t/2)} \left\{ 16n_r^2 \sum_f \sum_k \left[ \frac{1}{k^2 + M_{f_u}^2} \frac{1}{(k+p)^2 + M_{f_u}^2} \right] \right. \\
- \frac{4}{3} \sum_k \left[ \frac{1}{(k+p)^2 + M_S^2} \frac{k^2 + M_S^2}{(k^2 + M_S^2)(k^2 + M_G^2)} \right] \right\}.
\]

Here the total contribution from pairs of light states with mass \( M_u \) is proportional to

\[
(32n_r^2 - 2),
\]

which vanishes when \( n_r = 1/4 \). The negative norm threshold has neatly canceled the unphysical thresholds. The surviving thresholds are the physical \( \bar{K}K \) and taste singlet \( \pi\eta \).

D. Isosinglet \( f_0 \) correlator

In this case we use the isosinglet operator \((\rho_{uu,1} + \rho_{dd,1})/\sqrt{2}\). We have both quark-line-connected and quark-line-disconnected contributions

\[
B_{f_0}(p, \tau a) = B_{f_0,\text{conn}}(p, \tau a) + B_{f_0,\text{disc}}(p, \tau a)
\]

\[
B_{f_0,\text{conn}}(p, \tau a) = \frac{1}{2} \left[ B_{u,u;u,u,\text{conn}}(p, \tau a) + B_{d,d;d,d,\text{conn}}(p, \tau a) \right]
\]

\[
= - \sum_{\bar{x}} (-)^x \exp(i\vec{p} \cdot \bar{x} a) \left\{ \text{Tr}[M_u^{-1}(\bar{x}, \tau; 0, 0)M_u^{-1}(\bar{x}, \tau; 0, 0)] \right\}
\]

\[
B_{f_0,\text{disc}}(p, \tau a) = \frac{1}{2} \left[ B_{u,u;u,d,\text{disc}}(p, \tau a) + B_{u,u;d,d,\text{disc}}(p, \tau a) \right]
\]

\[
+ B_{d,d;u,u,\text{disc}}(p, \tau a) + B_{d,d;d,d,\text{disc}}(p, \tau a) \right]
\]

\[
= \frac{1}{2} \sum_{\bar{x}} \exp(i\vec{p} \cdot \bar{x} a) \left\{ \text{Tr}[M_u^{-1}(\bar{x}, \tau; \bar{x}, \tau)]\text{Tr}[M_u^{-1}(0, 0; 0, 0)] \right\}
\]

The weight of the disconnected part is \( n_f/4 \) for \( n_f = 2 \) degenerate flavors for the state. The connected part of the correlator is identical to the full \( a_0 \) correlator.

In terms of the meson fields, the bubble correlator is (for \( \tau a = tA \))

\[
B_{f_0}(p, tA) = \frac{\mu^2 A^6}{8} \sum \exp(i\vec{p} \cdot \vec{y} A) \sum_{r,s,f,b,r',s',f'}
\]

\[
\frac{1}{2} \left[ \left\{ \phi_{ur,fs}(t)\phi_{fs,ur}(t)\phi_{dr',fs'}(0)\phi_{fs',dr'}(0) \right\} \right.
\]

\[
+ \left\{ \phi_{ur,fs}(t)\phi_{fs,ur}(t)\phi_{dr',f's'}(0)\phi_{f's',dr'}(0) \right\} \right] \right.
\]

\[
+ \left\{ \phi_{dr,fs}(t)\phi_{fs,dr}(t)\phi_{dr',f's'}(0)\phi_{f's',dr'}(0) \right\}
\]

\[
+ \left\{ \phi_{dr,fs}(t)\phi_{fs,dr}(t)\phi_{dr',f's'}(0)\phi_{f's',dr'}(0) \right\}
\]

(58)
In momentum space the correlator becomes

\[
B_{f_0}(p) = \frac{\mu^2}{8A^2(L/2)^3(N_f/2)} \left\{ n_r^2 \sum_{f,b} \sum_{k} \left[ \frac{1}{k^2 + M_{f_{u,b}}^2} \right] \right. \\
+ 2n_r^2 \sum_{f,b} \sum_{k} \left[ \frac{1}{k^2 + M_{U_{u,b}}^2} \right] \\
- 4n_r \sum_{k} \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
- 4n_r \sum_{k} \left[ \frac{4\delta_A}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
- 4n_r \sum_{k} \left[ \frac{4\delta_V}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
+ 4n_r^2 \sum_{k} \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
+ 4n_r^2 \sum_{k} \left[ \frac{4\delta_A}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
\times \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
+ 4n_r^2 \sum_{k} \left[ \frac{4\delta_V}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
\left. \times \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \right\}.
\]

(59)

In terms of valence quark world lines the first five terms are quark-line connected and the last three are disconnected.

In the continuum limit we have

\[
B_{f_0}(p) = \frac{\mu^2}{8A^2(L/2)^3(N_f/2)} \left\{ 16n_r^2 \sum_{f,b} \sum_{k} \left[ \frac{1}{k^2 + M_{f_{u,b}}^2} \right] \\
+ 32n_r^2 \sum_{k} \left[ \frac{1}{k^2 + M_{U_{u,b}}^2} \right] \\
- 4n_r \sum_{k} \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
- 4n_r \sum_{k} \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \\
+ 4n_r^2 \sum_{k} \left[ \frac{1}{(k+p)^2 + M_{U_{u,b}}^2} \right] \right\}.
\]

(60)

The two-pion threshold \((p+k)^2 + M_{U}^2 = 0\) and \(k^2 + M_{U}^2 = 0\) has a weight proportional to

\[
(64n_r^2 - 1)\mu^2.
\]

(61)

When \(n_r = 1/4\) the weight is 3 (for three physical pion channels). Thus, once again, only physical thresholds survive the continuum limit.
E. Single-flavor staggered fermions

Single-flavor QCD has no Goldstone bosons. The low-lying pseudoscalar (call it the $\eta'$) is lifted by the anomaly. With the staggered fermion action, however, only the taste-singlet $\eta'$ is lifted by the anomaly. The other 15 members of the taste multiplet (call them $\eta$) remain light. The member with pseudoscalar taste is an exact Goldstone boson. Such a spectrum would seem to spell trouble for the rooted theory. It is interesting to examine the scalar meson ($f_0$) correlator to see how the corresponding rooted chiral theory heals itself in the continuum limit.

Call the single replicated flavor $u$. The connected meson correlator is as before [Eq. (45)]. We choose not to decouple the taste-singlet $\eta'$ in this case because it is the only physical meson. The disconnected correlator for the taste singlet is then

$$\langle \phi_{gs,fr}^I (-k) \phi_{f',g'}^{I'} (k) \rangle_{\text{disc}} = \frac{\delta_{r,s} \delta_{r',s'} \delta_{f,g} \delta_{f',g'}}{n_r} \left[ -\frac{1}{(k^2 + M_{UU}^2)} + \frac{1}{k^2 + M_{\eta'U}^2} \right].$$

Similarly, the disconnected correlators for the taste axial vector and taste vector can be written as

$$\langle \phi_{gs,fr}^A (-k) \phi_{f'^V}^{A'} (k) \rangle_{\text{disc}} = \frac{\delta_{r,s} \delta_{r',s'} \delta_{f,g} \delta_{f',g'}}{n_r} \left[ \frac{-\delta_A}{(k^2 + M_{UU}^2)(k^2 + M_{\eta'U}^2)} \right].$$

and ($A \to V$), where in this case $M_{\eta'U}^2 = M_{UU}^2 + n_r \delta_A$, and similarly for $M_{\eta'V}^2$.

With these changes the $f_0$ correlator becomes

$$B_{f_0}(p) = \frac{\mu^2}{8A^2(L/2)^3(N_c/2)} \left\{ 2n_r^2 \sum_b \sum_k \left[ \frac{1}{k^2 + M_{UU}^2} \frac{1}{(k+p)^2 + M_{UU}^2} \right] \right.$$  

$$- 4n_r \sum_k \left[ \frac{1}{(k+p)^2 + M_{UU}^2} \frac{1}{k^2 + M_{UU}^2} - \frac{1}{k^2 + M_{\eta'U}^2} \right]$$  

$$+ 4n_r \sum_k \left[ \frac{\delta_A}{((k+p)^2 + M_{UU}^2)(k^2 + M_{UU}^2)(k^2 + M_{\eta'U}^2)} \right]$$  

$$+ 4n_r \sum_k \left[ \frac{\delta_V}{((k+p)^2 + M_{UU}^2)(k^2 + M_{UU}^2)(k^2 + M_{\eta'U}^2)} \right]$$  

$$+ 2n_r^2 \sum_k \left[ \frac{\delta_A}{(k+p)^2 + M_{UU}^2} - \frac{1}{(k+p)^2 + M_{\eta'U}^2} \right] \frac{1}{n_r} \left[ \frac{1}{k^2 + M_{UU}^2} - \frac{1}{k^2 + M_{\eta'U}^2} \right]$$  

$$+ 2n_r^2 \sum_k \left[ \frac{\delta_A}{((k+p)^2 + M_{UU}^2)((k+p)^2 + M_{\eta'U}^2)(k^2 + M_{\eta'U}^2)} \right]$$  

$$+ 2n_r^2 \sum_k \left[ \frac{\delta_V}{((k+p)^2 + M_{UU}^2)((k+p)^2 + M_{\eta'U}^2)(k^2 + M_{\eta'U}^2)} \right] \right\}.$$
We note that a simplified version of our result (setting the discretization corrections from $\delta_A$ and $\delta_V$ to zero) was presented previously [2]. In the continuum limit the would-be Goldstone thresholds become degenerate with the negative-norm threshold, with a net weight proportional to

\[(32n_r^2 - 2)\]  

(65)

When $n_r = 1/4$, the would-be Goldstone bosons decouple from the $f_0$ correlator, leaving only the physical high-lying $\eta'\eta'$ channel.

IV. SIMULATIONS AND RESULTS

In this work we analyzed the 0.12 fm ensemble of 510 $24^3 \times 64$ gauge configurations generated in the presence of $2 + 1$ flavors of Asqtad improved staggered quarks with bare quark masses $am_{ud} = 0.005$ and $am_s = 0.05$ and bare gauge coupling $10/g^2 = 6.76$ [17].

We set valence quark masses equal to the sea quark masses. Table I gives the pseudoscalar masses used in our fits with the exception of the masses $\eta_A$, $\eta'_A$, $\eta_V$, $\eta'_V$. Those masses vary with the fit parameters $\delta_A$ and $\delta_V$.

For the light quark Dirac operator $M_u$, we measured the point-to-point quark-line connected correlator

\[
C_{\text{conn}}(\vec{p}, \tau) = \sum \langle - \rangle^x \cos(\vec{p} \cdot \vec{x}) \left\langle \text{Tr} M_u^{-1}(\vec{x}, \tau; 0, 0) M_u^{-1}(\vec{x}, \tau; 0, 0) \right\rangle
\]

(66)

and point-to-point quark-line disconnected correlator

\[
C_{\text{disc}}(\vec{p}, \tau) = \sum \langle - \rangle^x \cos(\vec{p} \cdot \vec{x}) \left\langle \text{Tr} M_u^{-1}(\vec{x}, \tau; \vec{x}, \tau) \text{Tr} M_u^{-1}(0, 0; 0, 0) \right\rangle.
\]

(67)

In the latter case we use noisy estimators based on random $Z(2)$ color vectors [26] $\eta_k$ for $k = 1, \ldots N = 200$:

\[
\text{Tr} M_u^{-1}(\vec{x}, \tau; \vec{x}, \tau) \text{Tr} M_u^{-1}(0, 0; 0, 0) \approx \frac{1}{N(N-1)} \sum_{k \neq k',y,y'} \tilde{\eta}_k(\vec{x}, \tau) M_u^{-1}(\vec{x}, \tau; y) \eta_k(y)
\]

(68)

\[
\times \tilde{\eta}_{k'}(0, 0) M_u^{-1}(0, 0; y') \eta_{k'}(y').
\]

In terms of these correlators the $a_0$ and $f_0$ correlators are

\[
C_{a_0}(\vec{p}, \tau) = C_{\text{conn}}(\vec{p}, \tau)
\]

\[
C_{f_0}(\vec{p}, \tau) = C_{\text{conn}}(\vec{p}, \tau) - \frac{1}{2} C_{\text{disc}}(\vec{p}, \tau)
\]

(69)
Correlators in each channel were measured at five momenta \( \vec{p} = (0, 0, 0) \), \( (1, 0, 0) \), \( (1, 1, 0) \), \( (1, 1, 1) \), and \( (2, 0, 0) \). All ten correlators were then fit to the following model

\[
C_{a_0}(\vec{p}, \tau) = C_{\text{meson}, a_0}(\vec{p}, \tau) + B_{a_0}(\vec{p}, \tau)
\]

\[
C_{f_0}(\vec{p}, \tau) = C_{\text{meson}, f_0}(\vec{p}, \tau) + B_{f_0}(\vec{p}, \tau)
\]

(70)

where

\[
C_{\text{meson}, a_0}(\vec{p}, \tau) = b_{a_0}(p) \exp[-E_{a_0}(p)\tau] + b_{\pi, A}(p)(-)^\tau \exp[-E_{\pi, A}(p)\tau] + \tau \rightarrow N_t - \tau
\]

(71)

\[
C_{\text{meson}, f_0}(\vec{p}, \tau) = c_0(p) + b_{f_0}(p) \exp[-E_{f_0}(p)\tau] + b_{\eta, A}(p)(-)^\tau \exp[-E_{\eta, A}(p)\tau] + \tau \rightarrow N_t - \tau.
\]

This fitting model adds explicit \( a_0 \) and \( f_0 \) poles, as well as the corresponding negative parity states, to the bubble contribution. Such states are outside the scope of the low order chiral Lagrangian in Eq. (2). Of course it is possible to enlarge the Lagrangian to include them [21]. Taste-breaking effects complicate this exercise. Moreover, we would need to introduce a variety of higher order chiral couplings, which are unlikely to be well constrained by our data. Therefore, we took the more modest approach and treated these additional terms empirically, keeping in mind the possibility of higher order chiral effects.

Our parameterization of the momentum dependence of the overlap factors \( b_j(p) \) requires some discussion. The \( a_0 \) and \( f_0 \) are produced through the scalar density with spin-taste assignment \( 1 \times 1 \). Thus at zeroth order in the \( a_0 - \pi - \eta \) coupling their contributions should be inversely proportional to their energies \( b_j(0) = 1/2E_j(p) \). At higher order an iteration of the bubble contribution alters the momentum dependence of the pole residue [21]. For present purposes we chose the empirical fitting form

\[
b_j(p) = b_{j0} + b_{j1}p^2.
\]

(72)

and adjusted the constants \( b_{j0} \) and \( b_{j1} \).

The negative parity states are the taste-axial-vector pion \( \pi_A \) and the taste-axial-vector \( \eta_A \). As staggered partners to the \( a_0 \) and \( f_0 \) they couple through axial vector currents with spin-taste assignment \( \gamma_0\gamma_5 \times \gamma_0\gamma_5 \), which contribute a factor of the energy to source and sink. Thus their bare momentum dependence should be proportional to their energies

\[
b_j(p) = b_jE(p).
\]

(73)

We kept this form, adjusting \( b_j \).
The constant $c_0(p)$ is zero for all momenta except $\vec{p} = (0,0,0)$, in which case it gives the vacuum-disconnected part of the $f_0$ correlator. There are eleven fit parameters for the meson terms alone, but the two negative parity masses were constrained tightly by priors: the $\pi_A$, to the previously measured value, and the $\eta_A$, to the same derived mass that we used in the bubble term.

The bubble terms $B_{a_0}$ and $B_{f_0}$ in the fitting function Eq (70) are given in momentum space by Eqs. (53) and (60). Their time-Fourier transforms yield $B_{a_0}(\vec{p}, \tau)$ and $B_{f_0}(\vec{p}, \tau)$ by applying the following identity term by term:

$$B(\vec{p}, \tau) \propto \frac{1}{A^2(N_t/2)^2} \sum_{p_0,k} \frac{e^{-ip_0t}}{(k^2 + M^2_1)((p-k)^2 + M^2_2)} = \sum_{\vec{k}} \frac{e^{-\sqrt{|\vec{k}|^2 + M^2_j}t}}{4E_1(\vec{k})E_2(\vec{k})} \tag{74}$$

where $E_j(\vec{k}) = \sqrt{|\vec{k}|^2 + M^2_j}$, and, as usual, $tA \equiv \tau a$. Thus, for example, the $\bar{K}K$ contribution to $B_{a_0}(\vec{p}, \tau)$ for taste $b$ is

$$\frac{\mu^2}{16L^3} \sum_{\vec{k}} \frac{e^{-[E_{Kb}(\vec{k})+E_{Kb}(\vec{k})]t}}{4E_{Kb}(\vec{k})E_{Kb}(\vec{k})} \tag{75}$$

The bubble terms $B_{a_0}(p, \tau)$ and $B_{f_0}(p, \tau)$ were parameterized by the three low energy couplings $\mu = m^2_\pi/(2m_\ell)$, $\delta_A = a^4\delta'_A$, and $\delta_V = a^4\delta'_A$ in the notation of Ref. [15]. They were allowed to vary to give the best fit. The taste multiplet masses in the bubble terms were fixed as noted above. The sum over intermediate momenta was cut off when the total energy of the two-body state exceeded $1.8/a$ or any momentum component exceeded $\pi/(3a)$. We determined that such a cut off gave acceptable accuracy for $\tau \geq 4$.

In summary, we fit all ten correlators with fourteen parameters, eleven of which were needed to parameterize the four explicit meson terms and three low energy couplings were needed for the bubble contribution. Through a prior, we constrained the value of $\delta_V$ to conform to previous fits to the pseudoscalar masses and decay constants [15], leaving only two of the low energy couplings to be adjusted independently. Our best fit gave $\chi^2/dof = 126/109$ (CL 0.13).

The fitted functional form is compared with the data in Figs 1–3.

Results of the fits are compared with results from fits to the meson masses and decay constants in Table II. The agreement is worse if we used the bare value $r_1\mu = 4.5$ from those fits, rather than the higher-order $m^2_\pi/(2m_\ell)$, suggesting, perhaps, that a higher order calculation of the bubble contribution might improve the agreement.
FIG. 1: Best fit to the $a_0$ correlator for five total cm momenta. The fitting range is indicated by points and fitted lines in red and blue (darker points and lines). Occasional points with negative central values are not plotted.

Our fit Meson masses and decays

<table>
<thead>
<tr>
<th></th>
<th>Our fit</th>
<th>Meson masses and decays</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 m^2/(2m_{u,d})$</td>
<td>7.3(1.6)</td>
<td>6.7</td>
</tr>
<tr>
<td>$\delta_V$</td>
<td>(prior)</td>
<td>$-0.016(23)$</td>
</tr>
<tr>
<td>$\delta_A$</td>
<td>$-0.056(10)$</td>
<td>$-0.040(6)$</td>
</tr>
</tbody>
</table>

TABLE II: Comparison of our fit parameters for the rS$\chi$PT low energy constants with results from [15]

The fitted masses of the $a_0$ and $f_0$ in units of the lattice spacing are 0.61(5) and 0.45(9), respectively.

V. SUMMARY AND CONCLUSIONS

We have derived the two-pseudoscalar-meson “bubble” contribution to the $f_0$ correlator in lowest order S$\chi$PT, thereby extending the result for the $a_0$ in reference [20]. We have used this model to fit simulation data for the point-to-point $a_0$ and $f_0$ correlators and found
FIG. 2: Best fit to the $f_0$ correlator for four total cm momenta.

FIG. 3: Best fit to the zero momentum $f_0$ correlator.

that best-fit values of the three chiral low energy couplings are in reasonable agreement with values previously obtained in fits to the light meson spectra and decay constants [15].

The two-meson bubble term in $S\chi$PT provides a useful illustration of the lattice artifacts
induced by the fourth-root approximation, since it involves quark loops coming from the fermion determinant. The artifacts include thresholds at unphysical energies and thresholds with negative weights. These are the same sorts of artifacts commonly observed with quenching or partial quenching. These contributions are clearly present in the $a_0$ and $f_0$ channels in our QCD simulation with the Asqtad action at $a = 0.12$ fm. We have found that they must be taken into account in a successful spectral analysis. Fortunately, $rS\chi$PT provides an explicit parameterization of their contributions for the interpolating operators we have chosen, thereby allowing a fit to simulation data with a manageable number of parameters. The $rS\chi$PT predicts further that these lattice artifacts disappear in the continuum limit, leaving only physical two-body thresholds. This result is in full accordance with the fourth-root analysis of Ref [2]. It will be interesting to see whether this expectation is borne out in numerical QCD simulations at smaller lattice spacing.

Acknowledgments

This work is supported in part by the US National Science Foundation, the US Department of Energy and Slovenian Ministry of Education, Science and Sport. We are grateful to the MILC Collaboration for the use of the Asqtad lattice ensemble. The analysis of these lattice files was carried out at the Utah Center for High Performance Computing.


