

Quantum Theory of Many-Particle Systems

W. H. Dickhoff

Wim

Department of Physics, Washington University,

St. Louis, MO 63130, USA

Notes for Physics 540

Fall Semester 2000

COURSE FORMAT

This course will include the following components :

1. Three lectures per week on Monday, Wednesday, and Friday from 10-11am in Compton 245 (attendance required)
2. One hour review/discussion meeting a week where problems are presented by the students (attendance required)
3. About one problem per class assigned to be presented by students during the discussion meeting
4. A few computer assignments
5. No exams
6. A paper summarizing a set of articles from the literature related to the material of the course is to be turned in before the last class. This paper must be in revtex format (used in the Physical Review journals) and should contain a proper set of references. Using the documentstyle [pra,aps] the paper should at least be 7 pages but not more than 10 pages long and may include equations but not detailed derivations (if necessary they can be included in an appendix). Half a page containing a proposal for the topic of the paper is due during the last class before Fall Break.
7. A 30-minute presentation on the material of the paper is required. Attendance at all talks by other students is also required. This talk should include a motivation, a discussion of the method of solution and experimental data (where appropriate), a discussion of the results, and a summary plus conclusions of the presented material. The use of overhead transparencies is recommended.
8. Classroom participation and discussion is mandatory.
9. Reading assignments should be completed before the next class.

COURSE GRADE

The course grade will contain the following contributions :

1. • Homework problems 20%
2. • Computer assignments 20%
3. • Paper 20%
4. • Presentation 20%
5. • Reading assignments 10%
6. • Everything else including classroom participation 10%

INTRODUCTION

This course is aimed at providing a systematic extension of a typical first-year graduate course in Quantum Mechanics. Tools and techniques will be developed to deal with nonrelativistic many-particle systems where quantum effects dominate. An important goal of this course is to provide a unified perspective on different fields of physics. Although details differ greatly when one studies atoms, molecules, electrons in solids, quantum liquids, nuclei, nuclear/neutron matter, or other systems, it is possible to use the same theoretical framework to develop relevant approximation schemes. The textbook by Fetter and Walecka [1] has a similar aim. It is available in the bookstore but its price and age make it very optional. Other relevant books are on reserve in the library [2] - [22]. A detailed outline of the course can be downloaded from the web page for the course at <http://www.physics.wustl.edu/~wimd/p540.html>. Many reading assignments are already indicated but others are preliminary.

Emphasis is on the Green's function or propagator method. This method is employed to rederive essential features of one- and two-body quantum mechanics including eigenvalue equations (discrete spectrum) and results relevant for scattering problems (continuum problem). Employing the occupation number representation (second quantization) the method is then further developed by utilizing Feynman diagrams and the equation of motion method, initially at zero and, subsequently, at finite temperature. Atomic systems, the electron gas, strongly correlated liquids including nuclear matter, neutron matter, and helium systems, as well as finite nuclei are used to illustrate various levels of sophistication needed in the description of these systems. These include the mean-field (Hartree-Fock) method, Random Phase Approximation (ring diagram summation), Binary Collision Approximation (summation of ladder diagrams), and further extensions. Results of these methods are, where possible, confronted with experimental data in the form of excitation spectra and transition probabilities or cross sections. Spectacular features of many-particle quantum mechanics in the form of Bose-Einstein condensation, superfluidity, and superconductivity will also be discussed.

The employed method focuses on the properties of one particle in the medium, as they are described by the single-particle propagator, and the characteristics of the interaction between particles in the medium which are contained in the two-particle propagator. These quantities yield most of the observables of quantum many-particle systems which are experimentally accessible. The analysis of the perturbation expansion of the one- and two-particle propagator generates the well-known Feynman diagrams which were first used in Quantum Field Theory. To emphasize the applicability of the method to any quantum many-particle system, the central quantity of the method, the single-particle propagator, is introduced in a general single-particle basis from the very beginning. In many texts considerable emphasis is placed on the coordinate and/or momentum space representations. In addition, the propagator description of the one- and two-particle problems is developed in considerable detail in order to illustrate the use of diagrammatic methods in a very simple context. The solution method of propagator equations for the one- and two-particle problems provides a blueprint for finding solutions of similar propagator equations in a many-particle setting.

The presentation is aimed at obtaining practical results in a way that is accessible to and also of relevance for experimentalists. Mathematical rigor is not emphasized. Instead, relatively straightforward manipulations are utilized (supplemented by notes in the form of handouts) with the intention to focus on the description of the physics of many-particle systems.

I. IDENTICAL PARTICLES

A. Some simple considerations

A system that is very dilute, *i.e.* has a very low density, is not expected to exhibit striking quantum features since the constituent particles are hardly aware of each other. One can make this statement quantitative by considering the energy levels for a particle in a box

$$\epsilon_{n_x, n_y, n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2), \quad (1)$$

where h is Planck's constant, m is the mass of the particle, L is the length of a side of the box, and the n_i can be any nonzero integer. McQuarrie [23] calculates the number of states below an energy E for the case where E is large enough that this number is essentially a continuous function of E (see p. 11). The result is given by

$$\Omega(E) = \frac{\pi}{6} \left(\frac{8mL^2E}{h^2} \right)^{3/2} = \frac{\pi}{6} \left(\frac{8mE}{h^2} \right)^{3/2} V, \quad (2)$$

where V is the volume. If we take the average energy of a particle to be $E = 3/2k_B T$, where k_B is Boltzmann's constant ($k_B = 1.38066 \times 10^{-23}$ J/K = 8.61733×10^{-5} eV/K) and T the temperature in Kelvin, one can check that the number of such states for a particle with $m = 10^{-25}$ kg, $L = 0.1m$, and $T = 300$ K is about 10^{30} . This means that the number of states available to a molecule under these conditions will be much larger than the number of molecules in the box under normal conditions of temperature and density. In fact, one does not expect indistinguishability to play a role when the number of particles N is much smaller than $\Omega(E)$. Using the above results this condition can be written as

$$\frac{\pi}{6} \left(\frac{12mk_B T}{h^2} \right)^{3/2} V \gg N. \quad (3)$$

Large mass, high temperature, and low density favor this condition. On p. 72 of Ref. [23] this quantity (divided by N) is compared to unity for a number of systems. Only for very light molecules at low temperature does one expect indistinguishability effects. For electrons in metals, however, the above condition is already dramatically violated at room temperature. Repeating this calculation for nucleons at a density of 0.17 nucleons per cubic fermi (the density observed in the interior of heavy nuclei), at $k_B T = 1$ MeV also demonstrates a severe violation of this condition.

Similar estimates are obtained by considering the thermal wavelength of a particle which is given by

$$\lambda_T = \left[\frac{h^2}{2\pi m k_B T} \right]^{1/2} \quad (4)$$

for a particle with mass m and energy $k_B T$. When λ_T^3 becomes comparable with the volume per particle (V/N) one expects the identity of particles to play a significant role.

B. Bosons and Fermions

Spin and statistics are related at the level of Quantum Field Theory (QFT). The Dirac equation for a spin 1/2 fermion cannot be quantized without insisting that the field operators obey anticommutation relations. These relations, in turn, lead to Fermi-Dirac statistics represented by the Pauli exclusion principle for fermions. Fermions comprise all fundamental particles with half-integer intrinsic spin. Composite systems with an odd number of fermion constituents also behave like fermions provided they are in the same quantum state of the composite system. Similarly, the quantization of Maxwell's equations without sources and currents, is only possible when commutation relations between the field operators are imposed leading to Bose-Einstein statistics. Bosons can be identified by integer intrinsic spin appropriate for fundamental particles like photons and gluons but composite systems containing an even number of fermions also qualify. A historical perspective on the development of quantum statistics can be found in Ref. [24]. Many interesting many-particle systems contain fermions as their basic constituents. Without recourse to QFT one can treat the consequences of the identity of spin 1/2 particles as a result which is based on experimental observation. Indeed, this is how Pauli in 1925 came to formulate his famous principle [25]. By analyzing experimental Zeeman spectra of atoms, he concluded that electrons in the atom could not occupy the same single-particle (sp)

quantum state. In order to deal with this experimental observation it is possible to postulate that quantum states which describe N identical fermions must be antisymmetrical upon interchange of any two of the particles. A similar postulate requiring symmetric states upon interchange pertains for quantum states for N identical bosons. Also here one can invoke experimental evidence to insist on symmetric states in order to account for Planck's radiation law [24]. It appears in general that only symmetric or antisymmetric many-particle states are encountered in nature. To implement these postulates and study their consequences, it is useful to repeat a few simple relations of sp quantum mechanics that also play an important role in many-particle quantum physics. A useful text on Quantum Mechanics for background material is the book by Sakurai [26]. Some of the following material can also be found in [14].

C. Antisymmetric and Symmetric Two-Particle States

A sp state is denoted in Dirac notation by a ket $|\alpha\rangle$, where α represents a complete set of sp quantum numbers. For a fermion, α can represent the position quantum numbers, \mathbf{r} , its spin (which is usually omitted), and the z -component of its spin along the z -axis. For a spinless boson the position quantum numbers, \mathbf{r} , may be chosen. Many other possible complete sets of quantum numbers can be considered. The most relevant choice usually depends on the specific problem considered and this is also true in a many-particle setting. This choice will be further discussed when the independent particle model is introduced. To keep the presentation general the notation $|\alpha\rangle$ will be used but when discussing specific examples, relevant choices of sp quantum numbers will be used.

The sp states form a complete set with respect to some complete set of commuting observables like the position operator, the total spin, and its third component. They are normalized such that

$$\langle\alpha|\beta\rangle = \delta_{\alpha,\beta} \quad (5)$$

where the Kronecker symbol is used to include the possibility of δ -function normalization for continuous quantum numbers. For eigenstates of the position operator one has for example

$$\langle\mathbf{r}, m_s|\mathbf{r}', m'_s\rangle = \delta(\mathbf{r} - \mathbf{r}')\delta_{m_s, m'_s} \quad (6)$$

for a spin 1/2 fermion. For a spinless boson

$$\langle\mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r} - \mathbf{r}') \quad (7)$$

is appropriate in this representation. The completeness of the sp states can be written as

$$\sum_{\alpha} |\alpha\rangle\langle\alpha| = 1 \quad (8)$$

and again in the case of continuous quantum numbers one must use an integration instead of a summation or a combination of both in the case of a mixed spectrum.

Complete sets of states for N particles can be obtained by forming product states. The essential ideas can already be elucidated by considering two particles. In this case the notation

$$|\alpha_1\alpha_2\rangle = |\alpha_1\rangle|\alpha_2\rangle \quad (9)$$

is introduced. The first ket on the right hand side of this equation refers to particle 1 and the second to particle 2. Such product states obey the following normalization condition

$$\langle\alpha_1\alpha_2|\alpha'_1\alpha'_2\rangle = \delta_{\alpha_1\alpha'_1}\delta_{\alpha_2\alpha'_2} \quad (10)$$

and completeness relation

$$\sum_{\alpha_1\alpha_2} |\alpha_1\alpha_2\rangle\langle\alpha_1\alpha_2| = 1. \quad (11)$$

While these product states are sufficient for two non-identical particles they do not incorporate the correct symmetry which is necessary to describe identical bosons or fermions. Indeed, for α_1 and α_2 different one has

$$|\alpha_2\alpha_1\rangle \neq |\alpha_1\alpha_2\rangle. \quad (12)$$

This represents a difficulty when one performs a measurement on this system when the two particles are identical. If one obtains α_1 for one particle and α_2 for the other, one does not know which of the states in Eq. (12) represents the two particles. In fact, the two particles could as well be described by

$$c_1|\alpha_1\alpha_2\rangle + c_2|\alpha_2\alpha_1\rangle \quad (13)$$

which leads to an identical set of eigenvalues when a measurement is performed. This degeneracy is known as exchange degeneracy. This exchange degeneracy presents a difficulty because a specification of the eigenvalues of a complete set of observables does not uniquely determine the state as one expects from the general postulates of quantum mechanics [27].

To display the way in which the antisymmetrization or symmetrization postulates avoid this difficulty it is convenient to introduce permutation operators. One defines the permutation operator P_{12} by

$$P_{12}|\alpha_1\alpha_2\rangle = |\alpha_2\alpha_1\rangle. \quad (14)$$

While introduced as interchanging the quantum numbers of the particles this operator can also be viewed as effectively interchanging the particles. Clearly,

$$P_{12} = P_{21} \text{ and } P_{12}^2 = 1. \quad (15)$$

Consider the hamiltonian of two identical particles :

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + V(|\mathbf{r}_1 - \mathbf{r}_2|). \quad (16)$$

The observables, like position and momentum, must appear symmetrically in the hamiltonian, as in the classical case. To study the action of P_{12} , consider an operator A_1 acting on particle 1

$$A_1|\alpha_1\alpha_2\rangle = a_1|\alpha_1\alpha_2\rangle \quad (17)$$

where a_1 is an eigenvalue of A_1 contained in the set of quantum numbers α_1 . Similarly, an operator A_2 acting on particle 2 will give

$$A_2|\alpha_1\alpha_2\rangle = a_2|\alpha_1\alpha_2\rangle. \quad (18)$$

Consider

$$\begin{aligned} P_{12}A_1|\alpha_1\alpha_2\rangle &= a_1P_{12}|\alpha_1\alpha_2\rangle \\ &= a_1|\alpha_2\alpha_1\rangle \\ &= A_2|\alpha_2\alpha_1\rangle \end{aligned} \quad (19)$$

and

$$\begin{aligned} P_{12}A_1|\alpha_1\alpha_2\rangle &= P_{12}A_1P_{12}^{-1}P_{12}|\alpha_1\alpha_2\rangle \\ &= P_{12}A_1P_{12}^{-1}|\alpha_2\alpha_1\rangle \end{aligned} \quad (20)$$

from which one deduces that

$$P_{12}A_1P_{12}^{-1} = A_2 \quad (21)$$

since this can be done for any state $|\alpha_1\alpha_2\rangle$. As a result one has

$$P_{12}HP_{12}^{-1} = H \quad (22)$$

or

$$[P_{12}, H] = 0 \quad (23)$$

implying that both operators can be diagonal simultaneously. Eigenkets of P_{12} are:

$$|\alpha_1\alpha_2\rangle_+ = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2\rangle + |\alpha_2\alpha_1\rangle\} \quad (24)$$

and

$$|\alpha_1\alpha_2\rangle_- = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2\rangle - |\alpha_2\alpha_1\rangle\}, \quad (25)$$

with eigenvalues +1 and -1, respectively. One can define the symmetrizer

$$S_{12} = \frac{1}{2}(1 + P_{12}) \quad (26)$$

and antisymmetrizer

$$A_{12} = \frac{1}{2}(1 - P_{12}) \quad (27)$$

which applied to any linear combination of $|\alpha_1\alpha_2\rangle$ and $|\alpha_2\alpha_1\rangle$ will automatically generate the symmetric or antisymmetric state, respectively. In the case of identical fermions, the Pauli exclusion principle results from the requirement that an N -particle state must be antisymmetrical upon interchange of any two particles. In the case of two particles this implies that the relevant state is the antisymmetrical one (leaving out the $-$ subscript):

$$|\alpha_1\alpha_2\rangle = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2\rangle - |\alpha_2\alpha_1\rangle\}. \quad (28)$$

This state vanishes when $\alpha_1 = \alpha_2$ incorporating Pauli's principle. The symmetric state for two bosons (Eq. (24)) is not yet properly normalized when $\alpha_1 = \alpha_2$ demonstrating the possibility that bosons can occupy the same sp quantum state. The properly normalized two-boson state is given by

$$|\alpha_1\alpha_2\rangle_S = \left[\frac{1}{2n_{\alpha_1}!n_{\alpha_2}!} \right]^{1/2} \{|\alpha_1\alpha_2\rangle + |\alpha_2\alpha_1\rangle\}, \quad (29)$$

where n_α denotes the number of particles in sp state α . Obviously

$$\sum_{\alpha} n_{\alpha} = 2 \quad (30)$$

in this case. From now on, the states for more than one particle which have angular brackets will denote the antisymmetric or symmetric states. It should also be noted that as required for fermions

$$|\alpha_2\alpha_1\rangle = -|\alpha_1\alpha_2\rangle \quad (31)$$

and both kets therefore represent the same physical state. Only one of these states should be counted when the completeness relation for two identical fermions is considered. In practice this can be accomplished by ordering the sp quantum numbers. Suppose one has a set of sp states labeled by discrete quantum numbers $|1\rangle, |2\rangle, |3\rangle, \dots$ etc. For two particles the completeness relation in terms of antisymmetric states then reads *e.g.*

$$\sum_{i < j} |ij\rangle \langle ij| = 1, \quad (32)$$

although one can use an unrestricted sum as well if one corrects for the number of equivalent states

$$\frac{1}{2!} \sum_{ij} |ij\rangle \langle ij| = 1. \quad (33)$$

For bosons the equivalent relations to Eqs. (32) and (33) are given by

$$\sum_{i \leq j} |ij\rangle \langle ij| = 1, \quad (34)$$

and

$$\sum_{ij} \frac{n_i!n_j!}{2!} |ij\rangle \langle ij| = 1. \quad (35)$$

D. Antisymmetric and Symmetric Many-Particle States

In dealing with N particles one can proceed similarly as in the case of two particles. Product states are denoted by

$$|\alpha_1\alpha_2\dots\alpha_N\rangle = |\alpha_1\rangle |\alpha_2\rangle \dots |\alpha_N\rangle \quad (36)$$

with

$$\begin{aligned} \langle\alpha_1\alpha_2\dots\alpha_N|\alpha'_1\alpha'_2\dots\alpha'_N\rangle &= \langle\alpha_1|\alpha'_1\rangle\langle\alpha_2|\alpha'_2\rangle\dots\langle\alpha_N|\alpha'_N\rangle \\ &= \delta_{\alpha_1,\alpha'_1}\delta_{\alpha_2,\alpha'_2}\dots\delta_{\alpha_N,\alpha'_N} \end{aligned} \quad (37)$$

and

$$\sum_{\alpha_1\alpha_2\dots\alpha_N} |\alpha_1\alpha_2\dots\alpha_N\rangle\langle\alpha_1\alpha_2\dots\alpha_N| = 1. \quad (38)$$

These product states do not incorporate the correct symmetry and one must therefore project out the linear combination which is symmetric or antisymmetric. This is accomplished by using the antisymmetrizer for N particle states containing fermions

$$A = \frac{1}{N!} \sum_p (-1)^p P, \quad (39)$$

where the sum is over all $N!$ permutations, P is a permutation operator for N particles, and the sign indicates whether the corresponding permutation is even or odd. A symmetrizer must be used for N identical bosons

$$S = \frac{1}{N!} \sum_p P. \quad (40)$$

Normalized antisymmetrical states are then given by

$$|\alpha_1\alpha_2\dots\alpha_N\rangle = \sqrt{N!} A |\alpha_1\alpha_2\dots\alpha_N\rangle \quad (41)$$

while for bosons one obtains

$$|\alpha_1\alpha_2\dots\alpha_N\rangle = \left[\frac{N!}{n_\alpha!n_{\alpha'}!\dots} \right]^{1/2} S |\alpha_1\alpha_2\dots\alpha_N\rangle \quad (42)$$

with

$$\sum_\alpha n_\alpha = N. \quad (43)$$

For three fermions for example, one has

$$|\alpha_1\alpha_2\alpha_3\rangle = 6^{-1/2} \{ |\alpha_1\alpha_2\alpha_3\rangle - |\alpha_2\alpha_1\alpha_3\rangle + |\alpha_2\alpha_3\alpha_1\rangle - |\alpha_3\alpha_2\alpha_1\rangle + |\alpha_3\alpha_1\alpha_2\rangle - |\alpha_1\alpha_3\alpha_2\rangle \}. \quad (44)$$

It should be clear now that antisymmetry upon interchange of any two particles is incorporated since

$$|\alpha_1\alpha_2\alpha_3\rangle = -|\alpha_2\alpha_1\alpha_3\rangle \quad (45)$$

and so on.

An example for three bosons is given by

$$\begin{aligned} |\alpha_1\alpha_1\alpha_2\rangle &= \left[\frac{1}{3!2!} \right]^{1/2} \{ |\alpha_1\alpha_1\alpha_2\rangle + |\alpha_1\alpha_1\alpha_2\rangle + |\alpha_1\alpha_2\alpha_1\rangle + |\alpha_2\alpha_1\alpha_1\rangle + |\alpha_2\alpha_1\alpha_1\rangle + |\alpha_1\alpha_2\alpha_1\rangle \} \\ &= \frac{1}{3^{1/2}} \{ |\alpha_1\alpha_1\alpha_2\rangle + |\alpha_1\alpha_2\alpha_1\rangle + |\alpha_2\alpha_1\alpha_1\rangle \}. \end{aligned} \quad (46)$$

Symmetry upon interchange of any two particles is again incorporated since

$$|\alpha_1\alpha_1\alpha_2\rangle = +|\alpha_1\alpha_2\alpha_1\rangle \quad (47)$$

and so on.

A consequence of this explicit construction of antisymmetric states for N particles is that no sp state can be occupied by two particles, *i.e.* the quantum numbers represented *e.g.* by α_1 cannot occur twice in any antisymmetric N -particle state. Pauli's exclusion principle is therefore incorporated. For a given antisymmetric N -particle state there are $N!$ physically equivalent states obtained by a permutation of the sp quantum numbers. This means that the completeness relation for N particles can be written either as

$$\sum_{\alpha_1\alpha_2\dots\alpha_N}^{\text{ordered}} |\alpha_1\alpha_2\dots\alpha_N\rangle \langle\alpha_1\alpha_2\dots\alpha_N| = 1 \quad (48)$$

or

$$\frac{1}{N!} \sum_{\alpha_1\alpha_2\dots\alpha_N} |\alpha_1\alpha_2\dots\alpha_N\rangle \langle\alpha_1\alpha_2\dots\alpha_N| = 1. \quad (49)$$

Normalization for states with ordered sp quantum numbers has the form

$$\begin{aligned} \langle\alpha_1\alpha_2\dots\alpha_N|\alpha'_1\alpha'_2\dots\alpha'_N\rangle &= \langle\alpha_1|\alpha'_1\rangle\langle\alpha_2|\alpha'_2\rangle\dots\langle\alpha_N|\alpha'_N\rangle \\ &= \delta_{\alpha_1,\alpha'_1}\delta_{\alpha_2,\alpha'_2}\dots\delta_{\alpha_N,\alpha'_N}, \end{aligned} \quad (50)$$

whereas if the sp states are not ordered the result is obtained in the form of a determinant

$$\langle\alpha_1\alpha_2\dots\alpha_N|\alpha'_1\alpha'_2\dots\alpha'_N\rangle = \begin{vmatrix} \langle\alpha_1|\alpha'_1\rangle & \langle\alpha_1|\alpha'_2\rangle & \dots & \langle\alpha_1|\alpha'_N\rangle \\ \langle\alpha_2|\alpha'_1\rangle & \langle\alpha_2|\alpha'_2\rangle & \dots & \langle\alpha_2|\alpha'_N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\alpha_N|\alpha'_1\rangle & \langle\alpha_N|\alpha'_2\rangle & \dots & \langle\alpha_N|\alpha'_N\rangle \end{vmatrix}. \quad (51)$$

The normalized N -particle wave function of an antisymmetric state is given by

$$\psi_{\alpha_1\alpha_2\dots\alpha_N}(x_1x_2\dots x_N) = (x_1x_2\dots x_N|\alpha_1\alpha_2\dots\alpha_N), \quad (52)$$

where

$$(x_1x_2\dots x_N| = \langle x_1|\langle x_2|\dots\langle x_N| \quad (53)$$

and $x_1 = \{\mathbf{r}_1, m_{s_1}\}$. Often this wave function is written in determinantal form

$$\psi_{\alpha_1\alpha_2\dots\alpha_N}(x_1x_2\dots x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \langle x_1|\alpha_1\rangle & \dots & \langle x_N|\alpha_1\rangle \\ \langle x_1|\alpha_2\rangle & \dots & \langle x_N|\alpha_2\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1|\alpha_N\rangle & \dots & \langle x_N|\alpha_N\rangle \end{vmatrix}. \quad (54)$$

Such a wave function is commonly called a Slater determinant. In practice it is very cumbersome to work with Slater determinants and calculate matrix elements of operators between many-particle states.

For N boson states there is no restriction on the occupation of a given sp state. In fact, all particles can occupy the same sp state! For a given symmetric N -particle state there are $N!$ physically equivalent states obtained by a permutation of the sp quantum numbers. In addition, one can have multiple occupation of a sp state. Such states should only be counted once in the completeness relation. In an unrestricted sum over quantum numbers for $N = 3$ all states

$$|\alpha_1\alpha_1\alpha_2\rangle = |\alpha_1\alpha_2\alpha_1\rangle = |\alpha_2\alpha_1\alpha_1\rangle \quad (55)$$

occur. The appropriate weighting of these states is obtained by including factorial factors $n_\alpha!$ in the completeness relation as follows

$$\sum_{\alpha_1\alpha_2\dots\alpha_N} \frac{n_{\alpha_1}!n_{\alpha_2}!\dots}{N!} |\alpha_1\alpha_2\dots\alpha_N\rangle \langle\alpha_1\alpha_2\dots\alpha_N| = 1. \quad (56)$$

When ordering of the sp states is considered no such factors need be included

$$\sum_{\alpha_1 \alpha_2 \dots \alpha_N}^{\text{ordered}} |\alpha_1 \alpha_2 \dots \alpha_N\rangle \langle \alpha_1 \alpha_2 \dots \alpha_N| = 1 \quad (57)$$

as in the case for fermions. Normalization for states with ordered sp quantum numbers has the form

$$\begin{aligned} \langle \alpha_1 \alpha_2 \dots \alpha_N | \alpha'_1 \alpha'_2 \dots \alpha'_N \rangle &= \langle \alpha_1 | \alpha'_1 \rangle \langle \alpha_2 | \alpha'_2 \rangle \dots \langle \alpha_N | \alpha'_N \rangle \\ &= \delta_{\alpha_1, \alpha'_1} \delta_{\alpha_2, \alpha'_2} \dots \delta_{\alpha_N, \alpha'_N}, \end{aligned} \quad (58)$$

whereas if the sp states are not ordered one has

$$\langle \alpha_1 \alpha_2 \dots \alpha_N | \alpha'_1 \alpha'_2 \dots \alpha'_N \rangle = \frac{1}{[n_{\alpha_1}! \dots n_{\alpha'_1}! \dots]^{1/2}} \sum_P \langle \alpha_1 | \alpha'_{p_1} \rangle \langle \alpha_2 | \alpha'_{p_2} \rangle \dots \langle \alpha_N | \alpha'_{p_N} \rangle. \quad (59)$$

The sum on the right-hand side is called a permanent. The normalized N -particle wave function of a symmetric state is also given by

$$\psi_{\alpha_1 \alpha_2 \dots \alpha_N}(x_1 x_2 \dots x_N) = (x_1 x_2 \dots x_N | \alpha_1 \alpha_2 \dots \alpha_N). \quad (60)$$

E. Some experimental consequences related to identical particles

Scattering experiments are an ideal tool to illustrate the consequences of dealing with identical particles. In the case of two particles that have identical mass and charge but can be distinguished in some other way, let's say their color being red or blue, a scattering experiment performed in the center of mass of these particles can have two distinguishable outcomes for the same scattering angle. If the red particle approaches in the z -direction and detectors that can distinguish red and blue are located in the direction θ (detector $D1$) and $\pi - \theta$ (detector $D2$) with the z -axis, the (quantummechanical) cross section for the red particle in $D1$ and the blue particle in $D2$ is given by

$$\frac{d\sigma}{d\Omega}(\text{red } D1, \text{blue } D2) = |f(\theta)|^2, \quad (61)$$

where $f(\theta)$ is the scattering amplitude [26]. The cross section for the red particle in $D2$ and the blue particle in $D1$ is given by

$$\frac{d\sigma}{d\Omega}(\text{red } D2, \text{blue } D1) = |f(\pi - \theta)|^2. \quad (62)$$

If the detectors are colorblind one cannot distinguish between these processes and the cross section for a count in $D1$ becomes the sum of the two probabilities

$$\frac{d\sigma}{d\Omega}(\text{count in } D1) = |f(\theta)|^2 + |f(\pi - \theta)|^2. \quad (63)$$

With identical bosons both processes cannot even in principle be distinguished. This implies that the probability amplitudes must be added (the wave function of the pair must also be symmetrical) before squaring to obtain the cross section which now reads

$$\frac{d\sigma}{d\Omega}(\text{bosons}) = |f(\theta) + f(\pi - \theta)|^2, \quad (64)$$

which now includes an interference term. The result of the interference is that at $\theta = \pi/2$ the cross section for bosons is twice that for distinguishable particles (but colorblind detectors). This prediction is confirmed by experiment. In the case of identical fermions one only obtains the interference when both particles have identical spin quantum numbers. In that case the cross section is given by

$$\frac{d\sigma}{d\Omega}(\text{fermions}) = |f(\theta) - f(\pi - \theta)|^2. \quad (65)$$

In this case no particles will be detected at all at $\theta = \pi/2$!

-
- [1] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, San Francisco, 1971).
- [2] D. Pines, *The Many-Body Problem* (Benjamin, Reading, MA, 1962), reissued in *Advanced Book Classics* series in 1997.
- [3] L.P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, Menlo Park CA, 1962), reissued in *Advanced Book Classics* series in 1989.
- [4] P.W. Anderson, *Concepts in Solids* (Benjamin, Reading MA, 1963).
- [5] D. Pines and P. Nozières, *The Theory of Quantum Liquids, Vol. I: Normal Fermi Liquids* (Benjamin, New York, NY, 1966).
- [6] A.B. Migdal, *Theory of Finite Fermi Systems and Applications* (Interscience, New York, 1967).
- [7] D.J. Thouless, *The Quantum Mechanics of Many-Body Systems* (Academic Press, New York, 1972).
- [8] G.E. Brown, *Many-Body Problems* (North-Holland, Amsterdam, 1972).
- [9] S. Doniach and E.H. Sondheimer, *Green's Functions for Solid State Physicists* (Benjamin, Reading, MA, 1974).
- [10] A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).
- [11] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon, Oxford, 1980).
- [12] J.R. Schrieffer, *Theory of Superconductivity* (Benjamin, Reading, MA, 1983 third printing).
- [13] P.W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin, Menlo Park CA, 1984).
- [14] J.P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge MA, 1986).
- [15] W.D. Kraeft, D. Kremp, W. Ebeling, and G. Röpke, *Quantum Statistics of Charged Particle Systems* (Plenum, New York, 1986).
- [16] D.S. Koltun and J.M. Eisenberg, *Quantum Mechanics of Many Degrees of Freedom* (Wiley, New York, 1988).
- [17] J.W. Negele and H. Orland, *Quantum Many-Particle Systems* (Benjamin, Redwood City CA, 1988).
- [18] G.D. Mahan, *Many-Particle Physics* (Plenum Press, New York, 1990) 2nd edition.
- [19] E.K.U. Gross, E. Runge, and O. Heinonen, *Many-Particle Theory* (Adam Hilger, Bristol, 1991).
- [20] R.D. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem* (Dover, New York, 1992).
- [21] P. Nozières and D. Pines, *The Theory of Quantum Liquids, Vol. II: Superfluid Bose Systems* (Addison-Wesley, Redwood City CA, 1990).
- [22] P. Nozières, *Theory of Interacting Fermi Systems* (Addison-Wesley, Reading, Massachusetts, 1997).
- [23] D.A. McQuarrie, *Statistical Mechanics* (Harper & Row, New York, 1976).
- [24] A. Pais, *Inward Bound* (Oxford University Press, Oxford, 1986).
- [25] W. Pauli, *Z. f. Phys.* **31**, 765 (1925).
- [26] J.J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading MA, 1994) revised edition.
- [27] P.A.M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, 4th ed., 1958).
- [28] I. Lindgren and J. Morrison, *Atomic Many-Body Theory* (Springer, New York, 1982).
- [29] M. Goeppert-Mayer, *Phys. Rev.* **75**, 1969 (1949).
- [30] O. Haxel, J.H.D. Jensen, and H.E. Suess, *Phys. Rev.* **75**, 1766 (1949).
- [31] H. Georgi, *Lie Algebras in Particle Physics* (Benjamin, Reading MA, 1982).
- [32] L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part 1, 3rd edition* (Pergamon, Oxford, 1980).
- [33] K. Huang, *Statistical Mechanics, 2nd edition* (Wiley, New York, 1987).
- [34] H. Yukawa, *Proc. Phys. Math. Soc. Japan* **17**, 48 (1935).
- [35] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980).
- [36] R.A. Aziz, V.P.S. Nain, J.C. Carly, W.J. Taylor, and G.T. McConville, *J. Chem. Phys.* **70**, 4330 (1979).
- [37] R.B. Wiringa, R.A. Smith, and T.L. Ainsworth, *Phys. Rev. C* **29**, 1207 (1984).
- [38] A. Griffin, D.W. Snoke, and S. Stringari, eds., *Bose-Einstein Condensation* (Cambridge University Press, Cambridge, 1995).
- [39] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
- [40] W. Ketterle, *Phys. Today* **52**, No. 12, 30 (1999).
- [41] K. Burnett, M. Edwards, and C.W. Clark, *Phys. Today* **52**, No. 12, 37 (1999).
- [42] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *Science* **269**, 198 (1995).
- [43] K.B. Davis *et al.*, *Phys. Rev. Lett.* **75**, 3969 (1995).
- [44] P. Sokol, in *Bose-Einstein Condensation*, edited by A. Griffin, D.W. Snoke, and S. Stringari (Cambridge University Press, Cambridge, 1995), p. 51.
- [45] A.F.G. Wyatt, *Nature* **391**, 56 (1998).
- [46] D.G. Fried *et al.*, *Phys. Rev. Lett.* **81**, 3811 (1998).
- [47] S. Chu, *Rev. Mod. Phys.* **70**, 685 (1998).
- [48] C. Cohen-Tannoudji, *Rev. Mod. Phys.* **70**, 707 (1998).

- [49] W.D. Phillips, *Rev. Mod. Phys.* **70**, 721 (1998).
- [50] C.C. Bradley, *et al.*, *Phys. Rev. Lett.* **75**, 1687 (1995).
- [51] B. DeMarco and D.S. Jin, *Science* **285**, 1703 (1999).
- [52] E.N. Economou, *Green's Functions in Quantum Physics, 2nd edition* (Springer, New York, 1983).
- [53] D.A. Butts and D.S. Rokhsar, *Phys. Rev. A* **55**, 4346 (1997).
- [54] K. Gottfried, *Quantum Mechanics* (Addison-Wesley, Redwood City, CA, 1989).
- [55] J.D. Jackson, *Classical Electrodynamics, 2nd edition* (Wiley, New York, 1975).
- [56] J.J. Sakurai, *Advanced Quantum Mechanics* (Benjamin, Menlo Park, CA, 1967).
- [57] M. Gell-Mann and F. Low, *Phys. Rev.* **84**, 350 (1951).
- [58] J. Goldstone, *Proc. Roy. Soc. (London)* **A239**, 267 (1957).
- [59] B.D. Day, *Rev. Mod. Phys.* **39**, 719 (1967).
- [60] H. Lehmann, *Nuovo Cim.* **11**, 342 (1954).