QUANTUM MECHANICS I (523)  
PROBLEM SET 10 (hand in December 4) 

(37) Calculate the following commutation relations: 

a) 

\[ [\ell_i, x_j] \]

b) 

\[ [\ell_i, p_j] \]

c) 

\[ \left[ p_i, \frac{1}{r} \right] \]

d) 

\[ \left[ p_i, \frac{x_j}{r} \right] \]

e) 

\[ [(\ell \times p)_i, p_j] \]

f) 

\[ \left[ (\ell \times p)_i, \frac{1}{r} \right], \]

where \( i \) and \( j \) correspond to \( x, y \) or \( z \), as usual.

(38) Define the operator 

\[ M = \frac{1}{2m} (p \times \ell - \ell \times p) - e^2 \frac{r}{r}. \]

a) Show that 

\[ [\ell_i, M_j] = i\hbar \epsilon_{ijk} M_k. \]

You should of course make use of the results of problem 37.
b) The Hamiltonian of the hydrogen atom

\[ H = \frac{p^2}{2m} - \frac{e^2}{r} \]

also commutes with \( \mathbf{M} \). Demonstrate this.

(39) Consider the Hamiltonian for the two-dimensional harmonic oscillator

\[ H_{x,y} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2. \]

Define

\[ \ell = xp_y - yp_x \]

and write \( \ell \) in terms of creation and annihilation operators of the corresponding one-dimensional harmonic oscillators. Show that \( \ell \) is a constant of motion. Besides the complete basis \( \{ |n_x, n_y \rangle \} \) one can choose a basis in which \( \ell \) is diagonal. Which other observable can be used to label these basis kets and make the basis complete? Argue your case.

(40) The three-dimensional harmonic oscillator with Hamiltonian

\[ H_{x,y,z} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2 z^2 \]

can also be diagonalized in two different basis sets: \( \{ |n_x, n_y, n_z \rangle \} \) or \( \{ |n, \ell, m \rangle \} \).

Consider the following linear combinations:

\[ |\alpha_1 \rangle = \frac{1}{\sqrt{2}} (|n_x = 0, n_y = 0, n_z = 0 \rangle + |n_x = 0, n_y = 0, n_z = 1 \rangle) \]

and

\[ |\alpha_2 \rangle = \frac{1}{\sqrt{2}} (|n_x = 1, n_y = 0, n_z = 0 \rangle - i |n_x = 0, n_y = 1, n_z = 0 \rangle). \]

Answer for each of these states the following questions: Does it correspond to

a) a stationary state?

b) an eigenstate of \( \ell^2 \)?

c) and eigenstate of \( \ell_z \)? Here it may be useful to employ the results of problem 39.

Give a short explanation or explicit calculation in each case.