QUANTUM MECHANICS II (524)
PROBLEM SET 8 (hand in March 26)

(29) Show that it is possible to obtain the following expression for the total cross section

\[ \sigma_{\text{tot}} \simeq \frac{m^2}{\pi \hbar^2} \int d^3r \int d^3r' \ V(r)V(r') \frac{\sin^2(p|r - r'|/\hbar)}{p^2|r - r'|^2} \]

in two different ways.

a) By integrating the differential cross section computed to first-order Born approximation.

b) By applying the optical theorem to the scattering amplitude at zero degrees in the second-order Born approximation.

Use the spherical symmetry of the total cross section and note that \( V \) is also assumed to be spherically symmetric.

(30) Consider a potential

\[ V = 0 \quad \text{for } r > R, \quad V = V_0 = \text{constant} \quad \text{for } r < R, \]

where \( V_0 \) may be positive or negative. Using the method of partial waves, show that for \( |V_0| \ll E = p^2/2m \) and \( pR/\hbar \ll 1 \) the differential cross section is isotropic and that the total cross section is given by

\[ \sigma_{\text{tot}} = \frac{16\pi m^2 V_0^2 R^6}{9 \hbar^4}. \]
Suppose that the energy is now raised slightly. Show that the angular
distribution can then be written as

\[ \frac{d\sigma}{d\Omega} = A + B \cos \theta. \]

Obtain an expression for \( A/B \).

(31) Prove that

\[ \frac{\hbar^2}{2m} \langle r \mid \frac{1}{E - H_0 + i\eta} \mid r' \rangle = -i \frac{p}{\hbar} \sum_{\ell} \sum_{m} Y_{\ell m}(\hat{\mathbf{r}}) Y_{\ell m}^*(\hat{\mathbf{r}}') j_\ell(pr_\ell/\hbar) h_\ell^{(1)}(pr_\ell/\hbar), \]

where \( r_<(r_>) \) stands for the smaller (larger) of \( r \) and \( r' \). Use the appendix
of Sakurai for definitions of Bessel and Hankel functions.

(32) For spherically symmetric potentials one can write the
Lippmann-Schwinger equation in the partial wave basis as follows

\[ |\Psi_{\ell m}^+\rangle = |\ell m\rangle + \frac{1}{E - H_0 + i\eta} V |\Psi_{\ell m}^+\rangle. \]

Use (31) to show that in the coordinate space basis this equation leads to
an integral equation for the radial wave function, \( \psi_\ell(p; r) \), as follows:

\[ \psi_\ell(p; r) = j_\ell(pr/\hbar) - \frac{2mip}{\hbar^3} \int_0^\infty \, dr' \, r'^2 \, j_\ell(pr_\ell/\hbar) h_\ell^{(1)}(pr_\ell/\hbar) V(r') \psi_\ell(p; r'). \]