QUANTUM MECHANICS I (523)
PROBLEM SET 5 (hand in October 9)

(17) Consider an electron in a static, uniform magnetic field with strength $B$
pointing in the $z$-direction. At $t = 0$ the electron is in an eigenstate of $S \cdot \hat{n}$ with
eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector, in the $xz$-plane making an angle $\beta$ with
the $z$-axis.

a) Determine the probability to obtain $\hbar/2$ for a measurement of $S_x$ as a function
of time.

b) Calculate the expectation value of $S_x$ as a function of time and make sure that
the result makes sense for both $\beta = 0$ and $\pi/2$.

(18) A two-state system has a Hamiltonian given by

$$H = h_{11} \langle 1 | 1 \rangle + h_{22} \langle 2 | 2 \rangle + h_{12} (\langle 1 | 2 \rangle + \langle 2 | 1 \rangle),$$

where $h_{11}, h_{22}$, and $h_{12}$ are real numbers, and $|1\rangle$ and $|2\rangle$ are eigenkets of some
other observable different from $H$.

a) Find the energy eigenkets and corresponding eigenvalues. Check that the limit
$h_{12} \to 0$ makes sense for your results.

b) Suppose by mistake the Hamiltonian was written as

$$H = h_{11} |1\rangle \langle 1| + h_{22} |2\rangle \langle 2| + h_{12} |1\rangle \langle 2|.$$

What problem will occur when time evolution is considered with this illegal
Hamiltonian? Illustrate this problem explicitly by considering the evolution of
some general initial state $|\psi(t_0)\rangle = \lambda_1 |1\rangle + \lambda_2 |2\rangle$ while assuming $h_{11} = h_{22} = 0$
for simplicity.

It is helpful not to use the results of part a) while solving part b).

(19) Consider a spinless particle in one dimension with a Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x).$$
a) Evaluate the commutator

\[ A = [H, x]. \]

Next evaluate

\[ [A, x]. \]

b) Use this result and appropriate completeness relations to show that

\[ \sum_i |\langle E_j | x | E_i \rangle|^2 (E_i - E_j) = \frac{\hbar^2}{2m}, \]

where the \{\ket{E_i}\} are energy eigenkets with corresponding eigenvalues \(E_i\).

(20) Consider the spin-precession discussed in class. This problem can also be solved in the Heisenberg picture. Use the Hamiltonian

\[ H = \omega S_z \]

to write the Heisenberg equations of motion for the time-dependent operators \(S_x(t), S_y(t),\) and \(S_z(t)\). Solve these equations of motion with initial conditions \(S_x(t = 0) = S_x(0), S_y(t = 0) = S_y(0),\) and \(S_z(t = 0) = S_z(0)\).