QUANTUM MECHANICS I (523)
PROBLEM SET 9 (hand in November 20)

(33) The wave function of a particle in a spherically symmetric potential \( V(r) \) is given by

\[
\psi(r) = (x + y + 3z)f(r)
\]

a) Is \( \psi \) an eigenfunction of \( \ell^2 \)? If yes, determine the \( \ell \)-value. If no, what are the possible \( \ell \)-values one may obtain upon measuring \( \ell^2 \)?

b) What are the probabilities for the particle to be found in various \( m_\ell \) states?

(34) A particle in a spherically symmetrical potential is known to be in an eigenstate of \( \ell^2 \) and \( \ell_z \) with eigenvalues \( \hbar^2 \ell(\ell + 1) \) and \( m_\ell \), respectively. Prove that the following expectation values w.r.t this state are satisfied:

\[
\langle \ell_x \rangle = \langle \ell_y \rangle = 0
\]

and

\[
\langle \ell_x^2 \rangle = \langle \ell_y^2 \rangle = \frac{[\ell(\ell + 1)\hbar^2 - m_\ell^2\hbar^2]}{2}.
\]

Try to interpret this result.

(35) Suppose a half-integer \( \ell \)-value, say 1/2, were allowed for orbital angular momentum. From

\[
\ell_+ Y_{1/2,1/2}(\theta, \phi) = 0,
\]

one may deduce

\[
Y_{1/2,1/2}(\theta, \phi) \propto \exp\{i\phi/2\} \sqrt{\sin \theta}.
\]

Try to construct \( Y_{1/2,-1/2} \) by

a) applying \( \ell_- \) to \( Y_{1/2,1/2} \) and

b) using

\[
\ell_- Y_{1/2,-1/2}(\theta, \phi) = 0.
\]
Show that these two procedures lead to contradictory results (lending support to the notion that half-integer \( \ell \)-values are not possible).

(36) Consider an orbital angular-momentum state \(|\ell = 2, m_\ell = 0\rangle\). Suppose this state is rotated by an angle \( \beta \) about the \( y \)-axis. Determine the probabilities for the rotated state to be found in \( m_\ell = 0, \pm 1, \) and \( \pm 2 \).